

On Optimal Separation of Eigenvalues for a Quasiperiodic Jacobi Matrix

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Abstract

We consider quasiperiodic Jacobi matrices of size N with analytic coefficients. We show that, in the positive Lyapunov exponent regime, after removing some small sets of energies and frequencies, any eigenvalue is separated from the rest of the spectrum by $N^{-1}(\log N)^{-p}$, with $p > 15$.

Keywords. eigenvalues, eigenfunctions, resonances, quasiperiodic Jacobi matrix, avalanche principle, large deviations

1 Introduction

It is known that one-dimensional quasiperiodic Schrödinger operators in the regime of positive Lyapunov exponent exhibit exponential localization of eigenfunctions (see for example [Bou05]). Can one develop an inverse spectral theory in such a regime? This is one of two major questions behind our work. The most studied case is the discrete single frequency case. Since the inverse spectral theory for the periodic case is well-understood, it seems very natural to try to understand how the regime of positive Lyapunov exponent plays out with the periodic approximation of the frequency via the standard convergent of its continued fraction. Obviously, the optimal estimate for the separation of the eigenvalues of the quasiperiodic operator on a finite interval is crucial for this kind of approach. This is the second major question behind this work. It is easy to figure out that the desired separation for the operator on the interval $[0, N-1]$, with appropriate N , is $\gtrsim N^{-1}(\log N)^{-p}$ with $p < 1$. Is this the correct estimate? A common sense argument suggests that outside of a small exceptional set of eigenvalues the estimate should be $\gtrsim o(N^{-1})$. What is known about this problem? Goldstein and Schlag [GS11] proved the estimate $\gtrsim \exp\left(-(\log N)^A\right)$, with $A \gg 1$, which is far from optimal. In this paper we improve the separation to $N^{-1}(\log N)^{-p}$, with $p > 15$. Moreover, we prove it

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for quasiperiodic Jacobi matrices. Our interest in the more general case is motivated by the fact that quasiperiodic Jacobi operators are necessary for the solution of the inverse spectral problem for discrete quasiperiodic operators of second order. We note that this setting is also needed for the study of the extended Harper's model, which corresponds to $a(x) = 2\cos(2\pi x)$, $b(x) = \lambda_1 e^{2\pi i(x-\omega/2)} + \lambda_2 + \lambda_3 e^{-2\pi i(x-\omega/2)}$ (see [JKS05, JM12]). At the same time we want to stress that the main result of this paper improves on the known result for the Schrödinger case and makes it much closer to the optimal one.

We consider the quasiperiodic Jacobi operator $H(x, \omega)$ defined on $l^2(\mathbb{Z})$ by

$$[H(x, \omega)\phi](k) = -b(x + (k+1)\omega)\phi(k+1) - \overline{b(x + k\omega)}\phi(k-1) + a(x + k\omega)\phi(k),$$

where $a : \mathbb{T} \rightarrow \mathbb{R}$, $b : \mathbb{T} \rightarrow \mathbb{C}$ ($\mathbb{T} := \mathbb{R}/\mathbb{Z}$) are real analytic functions, b is not identically zero, and $\omega \in \mathbb{T}_{c,\alpha}$ for some fixed $c \ll 1$, $\alpha > 1$, where

$$\mathbb{T}_{c,\alpha} := \left\{ \omega \in (0, 1) : \|n\omega\| \geq \frac{c}{n(\log n)^\alpha} \right\}.$$

The special case of the Schrödinger operator ($b = 1$) has been studied extensively (see [CFKS87, CL90]).

It is known that the Diophantine condition imposed on ω is generic, in the sense that $\text{mes}(\cup_{c>0} \mathbb{T}_{c,\alpha}) = 1$. This Diophantine condition, first used by Goldstein and Schlag [GS01], has the advantage of allowing one to prove stronger large deviations estimates (in the positive Lyapunov exponent case) than for general irrational frequencies. The use of large deviations estimates in the study of quasiperiodic Schrödinger operators was pioneered by Bourgain and Goldstein [BG00]. Initially these estimates were established for transfer matrices. More recently Goldstein and Schlag [GS08] proved a large deviations estimate for the entries of the transfer matrices (or equivalently for the determinants of the finite scale restrictions of the operator). This estimate is essential for our work, as it was for the developments in [GS08] and [GS11]. The technical details of extending the large deviations estimate for the entries to the Jacobi setting were dealt with in [BV12]. This reduces the cost of presenting our result in the more general Jacobi setting. Large deviations estimates in the quasiperiodic Jacobi case were also obtained in [JKS09, JM11, Tao12], but only for the transfer matrices.

We proceed by introducing the notation needed to state our main result. To motivate its statement we will first recall two results from [GS11].

It is known that a and b admit complex analytic extensions. We will assume that they both extend complex analytically to a set containing the closure of

$$\mathbb{H}_{\rho_0} := \{z \in \mathbb{C} : |\text{Im}z| < \rho_0\},$$

for some $\rho_0 > 0$. Let \tilde{b} denote the complex analytic extension of \bar{b} to \mathbb{H}_{ρ_0} .

We consider the finite Jacobi submatrix on $[0, N-1]$, denoted by $H^{(N)}(z, \omega)$, and defined by

$$\begin{bmatrix} a(z) & -b(z+\omega) & 0 & \dots & 0 \\ -\tilde{b}(z+\omega) & a(z+\omega) & -b(z+2\omega) & \dots & 0 \\ \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & -\tilde{b}(z+(N-1)\omega) & a(z+(N-1)\omega) \end{bmatrix}.$$

It is important for us to use \tilde{b} instead of \bar{b} , because we want the determinant to be complex analytic. More generally, we will denote the finite Jacobi submatrix on $\Lambda = [a, b]$ by $H_\Lambda(z, \omega)$. Let $E_j^{(N)}(z, \omega)$, and $\psi_j^{(N)}(z, \omega)$, $j = 1, \dots, N$ denote the eigenvalues and the l^2 -normalized eigenvectors of $H^{(N)}(z, \omega)$.

Let $L(\omega, E)$ be the Lyapunov exponent of the cocycle associated with $H(x, \omega)$. Our work deals with the case of the positive Lyapunov exponent regime. Namely, in this paper we assume that there exist intervals $\Omega^0 = (\omega', \omega'')$, $\mathcal{E}^0 = (E', E'')$ such that $L(\omega, E) > \gamma > 0$ for all $(\omega, E) \in \Omega^0 \times \mathcal{E}^0$.

We will be interested in the measure and complexity of sets $S \subset \mathbb{C}$. Writing $\text{mes}(S) \leq c$, $\text{compl}(S) \leq C$, will mean that there exists a set S' such that $S \subset S' \subset \mathbb{C}$ and $S' = \bigcup_{j=1}^K \mathcal{D}(z_j, r_j)$, with $K \leq C$, and $\text{mes}(S') \leq c$.

Goldstein and Schlag proved the following finite scale version of Anderson localization, in the Schrödinger case (see also [GS11, Lemma 6.4]). We give a restatement of [GS11, Corollary 9.10] adapted to our setting. Note that in this paper the constants implied by symbols such as \lesssim will only be absolute constants.

Proposition 1.1. ([GS11, Corollary 9.10]) *Given $A > 1$ there exists $N_0 = N_0(a, \gamma, \alpha, c, \mathcal{E}^0, A)$ such that for $N \geq N_0$ there exist $\Omega_N \subset \mathbb{T}$, $\mathcal{E}_{N, \omega} \subset \mathbb{R}$ with*

$$\begin{aligned} \text{mes}(\Omega_N) &\lesssim \exp\left(-(\log \log N)^A\right), \text{compl}(\Omega_N) \lesssim N^4, \\ \text{mes}(\mathcal{E}_{N, \omega}) &\lesssim \exp\left(-(\log \log N)^A\right), \text{compl}(\mathcal{E}_{N, \omega}) \lesssim N^4, \end{aligned}$$

satisfying the property that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_N$ and any $x \in \mathbb{T}$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N, \omega}$ then there exists $\nu_j^{(N)}(x, \omega) \in [0, N-1]$ such that if we let

$$\Lambda_j := \left[\nu_j^{(N)}(x, \omega) - l, \nu_j^{(N)}(x, \omega) + l \right] \cap [0, N-1], l = (\log N)^{4A},$$

we have that

$$\left| \psi_j^{(N)}(x, \omega; n) \right| \leq C \exp(-\gamma \text{dist}(n, \Lambda_j)/2) \quad (1.1)$$

for all $n \in [0, N-1]$.

We will call $\nu_j^{(N)}(x, \omega)$ localization centre, Λ_j localization window, and we say that $E_j^{(N)}(x, \omega)$ is localized when (1.1) holds. By using this localization result Goldstein and Schlag were able to obtain the following quantitative separation for the finite scale eigenvalues (see also [GS11, Proposition 7.1]). As with the previous Proposition, we give a restatement of [GS11, Proposition 10.1] adapted to our setting.

Proposition 1.2. ([GS11, Proposition 10.1]) *Given $0 < \delta < 1$ there exist large constants $N_0 = N_0(\delta, a, \gamma, \alpha, c, \mathcal{E}^0)$ and $A = A(\delta, a, \gamma, \alpha, c, \mathcal{E}^0)$ ($\delta A \gg 1$) such that for any $N \geq N_0$, and $l = (\log N)^A$ there exist Ω_N , $\mathcal{E}_{N, \omega}$ as in the previous Proposition such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_N$ and all $x \in \mathbb{T}$ one has*

$$\left| E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega) \right| > \exp(-l^\delta) \quad (1.2)$$

for all $j \neq k$ provided $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N, \omega}$.

Such separation results play a crucial role in [GS08] and [GS11]. It is well-known that $E_j^{(N)}(x, \omega)$ depends real analytically on x and ω , but we don't have a priori control on the radius of convergence. Part of the importance of having such separation results is that they give us control on the radius of convergence. More specifically, it can be seen that having the separation from (1.2), guarantees that the eigenvalue $E_j^{(N)}(\cdot, \cdot)$ remains simple on a polydisk $\mathcal{D}(x, c \exp(-l^\delta)) \times \mathcal{D}(\omega, c \exp(-l^\delta)/N)$, where c is an absolute constant. Hence we can guarantee that $E_j^{(N)}(\cdot, \cdot)$ is complex analytic on a polydisk of controlled size.

The separation achieved through (1.2) is much smaller than N^{-1} , which might be considered the optimal separation. The goal of our work is to improve the separation given by (1.2), in an attempt to come closer to the optimal separation. We now state our main result. A more precise formulation is given by Theorem 7.8.

Main Result. *Fix $p > 15$. There exist constants $N_0 = N_0(a, b, \rho_0, c, \alpha, \gamma, \mathcal{E}^0, p)$, $c_0 < 1$, such that for any $N \geq N_0$ there exists a set Ω_N , with*

$$\text{mes}(\Omega_N) \lesssim (\log \log N)^{-c_0}, \text{compl}(\Omega_N) \lesssim N^2 (\log N)^p,$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_N$ there exists a set $\mathcal{E}_{N, \omega}$, with

$$\text{mes}(\mathcal{E}_{N, \omega}) \lesssim (\log \log N)^{-c_0}, \text{compl}(\mathcal{E}_{N, \omega}) \lesssim N (\log N)^6,$$

such that for any $x \in \mathbb{T}$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N, \omega}$, for some j , then

$$\left| E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega) \right| \geq \frac{1}{N (\log N)^p},$$

for any $k \neq j$.

Remark. The above result is not about an empty set. It is known that

$$\text{mes}(\cup_{x \in \mathbb{T}} \text{spec}(H^{(N)}(x, \omega)) \cap \mathcal{E}^0) \rightarrow \text{mes}(\text{spec}(H(x, \omega)) \cap \mathcal{E}^0)$$

and that $\text{mes}(\text{spec}(H(x, \omega)) \cap \mathcal{E}^0) > 0$ (see [GS11, Proposition 13.1 (10), (11)]). Hence, even though the set $\mathcal{E}_{N, \omega}$ is quite large, the bulk of the spectral bands will be outside of it.

Unsurprisingly, improving the separation comes at the cost of an increase in size for the sets of bad frequencies and of bad energies. The improved complexity bound for the set of bad energies is crucial, as we shall soon see. Our method of proving the main result doesn't directly give us a complexity bound for Ω_N . The stated bound follows from the stability of the separation under perturbation in ω , and thus reflects the fact that the separation is less stable under perturbation when p is larger.

We will obtain our improved separation by first proving an appropriate finite scale localization result. The known approach for obtaining localization at scale N is to first eliminate resonances at a smaller scale l . This goes back to Sinai's paper [Sin87]. Informally speaking, resonances occur when the spectra of $H_{\Lambda_1}(x, \omega)$ and $H_{\Lambda_2}(x, \omega)$ are "too close", for two "far away" intervals of length l , $\Lambda_1, \Lambda_2 \subset [0, N-1]$. Specifically, in our case,

eliminating resonances on $[0, N-1]$ at scale l amounts to having the following: there exist constants σ_N , Q_N , and a set $\Omega_N \subset \mathbb{T}$, with the property that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ there exists $\mathcal{E}_{N,\omega} \subset \mathbb{R}$ such that for any $x \in \mathbb{T}$ and any integer m , $Q_N \leq |m| \leq N$, we have

$$\text{dist}(\mathcal{E}^0 \cap \text{spec}(H^{(l)}(x, \omega) \setminus \mathcal{E}_{N,\omega}), \text{spec}(H^{(l)}(x + m\omega, \omega))) \geq \sigma_N. \quad (1.3)$$

This condition can be reformulated to hold for all energies in \mathcal{E}^0 at the cost of removing a set of bad phases. However, our improvement of separation comes at the cost of also losing control over the set of bad phases, we just have control on the corresponding set of bad energies. Given such an elimination of resonances, one can prove a localization result in the spirit of Proposition 1.1, with the size of the localization window proportional to Q_N (see Theorem 3.4). After establishing localization one can obtain a separation of the eigenvalues at scale N by $\exp(-CQ_N)$ (see Proposition 4.3). Up to this point our strategy is the one employed by Goldstein and Schlag for the Schrödinger case (see [GS08], [GS11]). We will always have $\exp(-CQ_N) \ll \sigma_N$, for the concrete values of σ_N and Q_N that we use. Using a bootstrapping argument we show that the separation can be improved to $\sigma_N/2$ (see Theorem 4.4). Note that this can be done only if one is able to “fatten” the set of bad energies $\mathcal{E}_{N,\omega}$ by σ_N . For example, this suggests that the best separation that could be obtained through Proposition 1.1 is by N^{-4+} . So, our strategy for obtaining a sharper separation is to improve the elimination of resonances.

To eliminate resonances we will consider for fixed j, k, m , the sets of (x, ω) for which

$$\left| E_j^{(l)}(x, \omega) - E_k^{(l)}(x + m\omega, \omega) \right| < \sigma_N. \quad (1.4)$$

We will need to show that the union over j, k, m is small (provided $|m|$ is large enough). Goldstein and Schlag approached this problem by using resultants. Let $f_N^a(z, \omega, E) := \det[H^{(N)}(z, \omega) - E]$. The resultant of $f_l^a(x, \omega, E)$ and $f_l^a(x + m\omega, \omega, E)$ is a polynomial $R(x, \omega, E)$ with the property that it vanishes if E is a zero for both determinants. Strictly speaking, to define R , one needs to first use the Weierstrass Preparation Theorem to factorize the two determinants. For more details see [GS11, Section 5]. The idea behind considering R is that one can use Cartan’s estimate (see Lemma 2.9) to eliminate the set where $\log|R|$ is too small, and hence remove sets corresponding to (1.4).

Our approach is based on considering only the parts of the graphs of the eigenvalues where the slopes are “good”, i.e. bounded away from zero. We will be able to control the size of the sets where we have (1.4), by using the following simple observations. Let $g(x, \omega) = E_j^{(l)}(x, \omega) - E_k^{(l)}(x + m\omega, \omega)$. If $\left| \partial_x E_k^{(l)}(x + m\omega, \omega) \right| > \tau$, for some $\tau > 0$, it can be seen that $|\partial_\omega g(x, \omega)| \gtrsim m\tau$, for m large enough. If for some fixed x and some interval I we have $|g(x, \omega)| < \sigma_N$ and $|\partial_\omega g(x, \omega)| \gtrsim m\tau$ for all $\omega \in I$, then the length of I is $\lesssim \sigma_N(m\tau)^{-1}$. Our main problem will be to control the number of such intervals I . Similar considerations are used by Goldstein and Schlag for the elimination of the so called triple resonances (see [GS11, Section 14]). To implement our ideas, one can be tempted to first try to eliminate (x, ω) for which $\left| \partial_x E_k^{(l)}(x + m\omega, \omega) \right| \leq \tau$. Doing this would only yield separation by at most N^{-2} , due to the dependence on m of the set corresponding to the “good” slopes. Instead we will eliminate (x, ω) for

which $\left| \partial_x E_j^{(l)}(x, \omega) \right| \leq \tau$. More precisely we will proceed as follows. Using a Sard-type argument it is possible to show that for fixed ω and $\tau > 0$ we can find a small set $\mathcal{E}_{l, \omega}(\tau)$ such that for any $x \in \mathbb{T}$, if $E_j^{(l)}(x, \omega) \notin \mathcal{E}_{l, \omega}(\tau)$, then $\left| \partial_x E_j^{(l)}(x, \omega) \right| > \tau$. Let $\tilde{\mathcal{E}}_{l, \omega}(\tau) := \{E : \text{dist}(E, \mathcal{E}_{l, \omega}) < \sigma_N\}$. We have that for any $x \in \mathbb{T}$, if $E_j^{(l)}(x, \omega) \notin \tilde{\mathcal{E}}_{l, \omega}(\tau)$ and (1.4) holds, then $\left| \partial_x E_k^{(l)}(x + m\omega, \omega) \right| > \tau$. We stress the fact that the previous statement holds for any $x \in \mathbb{T}$, and thus by fattening the set of bad energies we were able to circumvent one summation over m , which ultimately will allow us to get the improved separation. We still have to control the complexity of the set of ω 's such that $|g(x, \omega)| < \sigma_N$ and $E_j^{(l)}(x, \omega) \notin \tilde{\mathcal{E}}_{l, \omega}(\tau)$. It is not clear how to do this directly. Instead, we will tackle this problem by working on small intervals I_ω (of controlled size) around ω on which we have some stability of the “good” slopes, that is, such that there exists a small set $\mathcal{E}_{l, I_\omega}(\tau)$ with the property that if $E_j^{(l)}(x, \omega') \notin \mathcal{E}_{l, I_\omega}(\tau)$, $\omega' \in I_\omega$ then $|\partial_x E_j(x, \omega')| > \tau$. In this setting we will need to control the complexity of the set of frequencies $\omega' \in I_\omega$ such that $|g(x, \omega')| < \sigma_N$ and $E_j^{(l)}(x, \omega') \notin \tilde{\mathcal{E}}_{l, I_\omega}(\tau)$. This can be achieved by using Bézout's Theorem, in the case when the eigenvalues are algebraic functions (in this case a and b are trigonometric polynomials). The general result will follow through approximation.

For the stability of the “good” slopes under perturbations in ω we need the following type of estimate

$$\left| \partial_x E_j^{(l)}(x, \omega) - \partial_x E_j^{(l)}(x, \omega') \right| \leq C|\omega - \omega'|.$$

This can be easily obtained by using Cauchy's Formula, provided we have control on the size of the polydisk to which $E_j^{(l)}$ extends complex analytically. As we already discussed, such information can be obtained from a separation result. In the Schrödinger case we have the “a priori” separation via resultants. We will need to prove that this separation also holds in the Jacobi case.

Next we give a brief overview of the article. In Section 2 we will introduce some more notation, review the basic results needed for our work, and deduce some useful consequences of these results. In Section 3 and Section 4 we establish localization and separation assuming that we have elimination of resonances, of the type (1.3), with undetermined σ_N and Q_N (subject to some constraints). Next, in Section 5, we obtain the elimination of resonances via resultants and the corresponding localization and separation results. In Section 6 we prove our elimination of resonances via slopes in an abstract setting. The reason for choosing the abstract setting is twofold. First, it makes it straightforward to obtain elimination with different values of the parameters. We will need to apply the abstract elimination twice to achieve our stated separation. Second, we want to emphasize the fact that at its heart our argument is about algebraic functions, and not specifically about eigenvalues. In Section 7 we will obtain our main result. Finally, in the Appendix we give the details needed for some of the results stated in Section 2.

2 Preliminaries

In this section we present the basic tools that we will be using and we deduce some useful consequences. We refer to [GS11, Section 2] for the Schrödinger case of these results.

We proceed by introducing some notation. For ϕ satisfying the difference equation $H(z, \omega)\phi = E\phi$ let M_N be the N -step transfer matrix such that

$$\begin{bmatrix} \phi(N) \\ \phi(N-1) \end{bmatrix} = M_N \begin{bmatrix} \phi(0) \\ \phi(-1) \end{bmatrix}, N \geq 1.$$

We have

$$M_N(z, \omega, E) = \prod_{j=N-1}^0 \left(\frac{1}{b(z + (j+1)\omega)} \begin{bmatrix} a(z + j\omega) - E & -\tilde{b}(z + j\omega) \\ b(z + (j+1)\omega) & 0 \end{bmatrix} \right),$$

for z such that $\prod_{j=1}^N b(z + j\omega) \neq 0$. We also consider the following two matrices associated with M_N :

$$M_N^a(z, \omega, E) = \left(\prod_{j=1}^n b(z + j\omega) \right) M_N(z, \omega, E) \quad (2.1)$$

and

$$M_N^u(z, \omega, E) = \frac{1}{\sqrt{|\det M_N(z, \omega, E)|}} M_N(z, \omega, E).$$

A fundamental property of M_N^a is that its entries can be written in terms of the determinant $f_N^a(z, \omega, E)$ defined in the introduction:

$$M_N^a(z, \omega, E) = \begin{bmatrix} f_N^a(z, \omega, E) & -\tilde{b}(z) f_{N-1}^a(z + \omega, \omega, E) \\ b(z + N\omega) f_{N-1}^a(z, \omega, E) & -\tilde{b}(z) b(z + N\omega) f_{N-2}^a(z + \omega, \omega, E) \end{bmatrix} \quad (2.2)$$

(see [Tes00, Chapter 1], where such relations are deduced in a detailed manner). Let $f_N^u(z, \omega, E)$ be such that

$$M_N^u(z, \omega, E) = \begin{bmatrix} f_N^u(z, \omega, E) & \star \\ \star & \star \end{bmatrix}$$

($f_N^u(z, \omega, E)$ is the determinant of an appropriately modified Hamiltonian). Based on the definitions, it is straightforward to check that

$$\log \|M_N^u(z, \omega, E)\| = -\frac{1}{2} \left(\tilde{S}_N(z, \omega) + S_N(z + \omega, \omega) \right) + \log \|M_N^a(z, \omega, E)\|, \quad (2.3)$$

where $S_N(z, \omega) = \sum_{k=0}^{N-1} \log |b(z + k\omega)|$ and $\tilde{S}_N(z, \omega) = \sum_{k=0}^{N-1} \log |\tilde{b}(z + k\omega)|$. Note that $S_N(x, \omega) = \tilde{S}_N(x, \omega)$ for $x \in \mathbb{T}$. For $y \in (-\rho_0, \rho_0)$ we let

$$L_N(y, \omega, E) = \frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x + iy, \omega, E)\| dx,$$

$$L(y, \omega, E) = \lim_{N \rightarrow \infty} L_N(y, \omega, E) = \inf_{N \geq 1} L_N(y, \omega, E).$$

We also consider the quantities L_N^a, L_N^u, L^a, L^u which are defined analogously. Furthermore let $D(y) = \int_{\mathbb{T}} \log |b(x+iy)| dx$. When $y = 0$ we omit the y argument, so for example we write $L(\omega, E)$ instead of $L(0, \omega, E)$. It is straightforward to see that $L_N^u(\omega, E) = L_N(\omega, E)$ and hence $L^u(\omega, E) = L(\omega, E)$. Based on (2.3) it is easy to conclude that

$$L(\omega, E) = -D + L^a(\omega, E). \quad (2.4)$$

For a discussion of the objects and quantities introduced above see [BV12, Section 2]. We note that in [BV12] it was more convenient to identify \mathbb{T} with the unit circle in \mathbb{C} . So for example a and b are considered to be defined on an annulus \mathcal{A}_{ρ_0} . However, it is trivial to switch between our setting and that of [BV12].

In what follows we will keep track of the dependence of the various constants on the parameters of our problem. In order to simplify the notation we won't always record the dependence on ρ_0 . Dependence on any quantity is such that if the quantity takes values in a compact set, then the constant can be chosen uniformly with respect to that quantity. We will use E^0 to denote the quantity $\sup\{|E| : E \in \mathcal{E}^0\}$. We denote by $\|\cdot\|_{\infty}$ the L^{∞} norm on \mathbb{H}_{ρ_0} and we let $\|b\|_* = \|b\|_{\infty} + \max_{y \in [-\rho_0, \rho_0]} |D(y)|$. Note that, unless otherwise stated, the constants in different results are different. Furthermore, in this paper the constants implied by symbols such as \lesssim will only be absolute constants.

The following form of the large deviations estimate for the determinants follows from [BV12, Proposition 4.10]. We give a detailed discussion in the Appendix. Note that in the Appendix we also give a different proof of one of the results [BV12], which allows us to remove one of the quantities on which the constants from [BV12] depended.

Proposition 2.1. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exist constants $N_0 = N_0(\|a\|_{\infty}, \|b\|_*, |E|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$, and $C_1 = C_1(\|a\|_{\infty}, \|b\|_*, |E|, c, \alpha, \gamma)$ such that for every integer $N \geq N_0$ and any $H > 0$ we have*

$$\text{mes} \left\{ x \in \mathbb{T} : |\log |f_N^a(x, \omega, E)| - NL^a(\omega, E)| > H(\log N)^{C_0} \right\} \leq C_1 \exp(-H).$$

Next we recall a uniform upper bound for the transfer matrix. The following is a restatement of [BV12, Proposition 3.14]. See the appendix for a discussion of this result and of the consequences that follow.

Proposition 2.2. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exist constants $C_0 = C_0(\alpha)$ and $C_1 = C_1(\|a\|_{\infty}, \|b\|_*, |E|, c, \alpha, \gamma)$ such that for any integer $N > 1$ we have*

$$\sup_{x \in \mathbb{T}} \log \|M_N^a(x, \omega, E)\| \leq NL^a(\omega, E) + C_1(\log N)^{C_0}.$$

Note that $\log |f_N^a(z, \omega, E)| \leq \log \|M_N^a(z, \omega, E)\|$, so this uniform upper bound also applies for the determinants f_N^a . Next we state two useful consequences of the uniform upper bound from Proposition 2.2. See the Appendix for the proofs.

Corollary 2.3. *Let $(\omega_0, E_0) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega_0, E_0) > \gamma > 0$. There exist constants $N_0 = (\|a\|_{\infty}, \|b\|_*, |E_0|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$, and $C_1 = C_1(\|a\|_{\infty}, \|b\|_*, |E_0|, c, \alpha, \gamma)$ such that for $N \geq N_0$ we have*

$$\begin{aligned} \sup \left\{ \log \|M_N^a(x+iy, \omega, E)\| : x \in \mathbb{T}, |E - E_0|, |\omega - \omega_0| \leq N^{-C_1}, |y| \leq N^{-1} \right\} \\ \leq NL^a(\omega_0, E_0) + (\log N)^{C_0}. \end{aligned}$$

Corollary 2.4. *Let $x_0 \in \mathbb{T}$ and $(\omega_0, E_0) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega_0, E_0) > \gamma > 0$. There exist constants $N_0 = (\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$, and $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$ such that for $N \geq N_0$ we have*

$$\begin{aligned} & \|M_N^a(x + iy, \omega, E) - M_N^a(x_0, \omega_0, E_0)\| \leq \\ & (|E - E_0| + |\omega - \omega_0| + |x - x_0| + |y|) \exp\left(NL^a(\omega_0, E_0) + (\log N)^{C_0}\right) \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & |\log|f_N^a(x + iy, \omega, E)| - \log|f_N^a(x_0, \omega_0, E_0)|| \leq \\ & (|E - E_0| + |\omega - \omega_0| + |x - x_0| + |y|) \frac{\exp\left(NL^a(\omega_0, E_0) + (\log N)^{C_0}\right)}{|f_N^a(x_0, \omega_0, E_0)|}, \end{aligned} \quad (2.6)$$

provided $|E - E_0|, |\omega - \omega_0|, |x - x_0| \leq N^{-C_1}$, $|y| \leq N^{-1}$, and that the right-hand side of (2.6) is less than $1/2$.

We will also need a version of Corollary 2.3 for S_N and \tilde{S}_N . See the Appendix for a proof.

Lemma 2.5. *There exist constants $C_0 = C_0(\alpha)$, $C_1 = C_1(\|b\|_*, c, \alpha)$ such that for every $N > 1$ we have*

$$\sup\{S_N(x + iy, \omega) : x \in \mathbb{T}, |y| \leq N^{-1}\} \leq ND + C_1(\log N)^{C_0}$$

and

$$\sup\{\tilde{S}_N(x + iy, \omega) : x \in \mathbb{T}, |y| \leq N^{-1}\} \leq ND + C_1(\log N)^{C_0}.$$

Next we recall the Avalanche Principle and show how to apply it to the determinants f_N^a .

Proposition 2.6. (*[GS08, Proposition 3.3]*) *Let A_1, \dots, A_n , $n \geq 2$, be a sequence of 2×2 matrices. If*

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1, \quad (2.7)$$

$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n, \quad (2.8)$$

and

$$\max_{1 \leq j < n} (\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|) < \frac{1}{2} \log \mu \quad (2.9)$$

then

$$\left| \log \|A_n \dots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C_0 \frac{n}{\mu}$$

with some absolute constant C_0 .

Corollary 2.7. *Let $z \in \mathbb{H}_{N-1}$, $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$, and let C_0 be as in Proposition 2.1. Let l_j , $j = 1, \dots, m$, be positive integers such that $l \leq l_j \leq 3l$, $j = 1, \dots, m$, with l a real number such that $l > 2m/\gamma$, and let $s_k = \sum_{j < k} l_j$ (note that $s_1 = 0$). Assume that there exists $H \in (0, l(\log l)^{-2C_0})$ such that*

$$\log \left| f_{l_j}^a(z + s_j \omega, \omega, E) \right| > l_j L^a(\omega, E) - H(\log l_j)^{C_0}, j = 1, \dots, m,$$

$$\log \left| f_{l_j + l_{j+1}}^a(z + s_j \omega, \omega, E) \right| > (l_j + l_{j+1}) L^a(\omega, E) - H(\log(l_j + l_{j+1}))^{C_0},$$

$j = 1, \dots, m-1$. There exists a constant $l_0 = l_0(\|a\|_\infty, \|b\|_, |E|, c, \alpha, \gamma)$ such that if $l \geq l_0$ then*

$$\left| \log \left| f_{s_{m+1}}^a(z, \omega, E) \right| + \sum_{j=2}^{m-1} \log \|A_j^a(z)\| - \sum_{j=1}^{m-1} \log \|A_{j+1}^a(z) A_j^a(z)\| \right| \lesssim m \exp\left(-\frac{\gamma}{2}l\right),$$

where

$$A_1^a(z) = A_1^a(z, \omega, E) = M_{l_1}^a(z, \omega, E) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_m^a(z) = A_m^a(z, \omega, E) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{l_m}^a(z + s_m \omega, \omega, E),$$

and $A_j^a(z) = A_j^a(z, \omega, E) = M_{l_j}^a(z + s_j \omega, \omega, E)$, $j = 2, \dots, m-1$.

Proof. Note that $\log |f_{s_{m+1}}^a(z)| = \log \left\| \prod_{j=m}^1 A_j^a(z) \right\|$. Essentially, the conclusion follows by applying the Avalanche Principle. This is straightforward in the Schrödinger case. The Jacobi case is slightly more complicated because the matrices A_j^a don't necessarily satisfy (2.7). Let A_j^u be defined analogously to A_j^a (using M_l^u instead of M_l^a). The matrices A_j^u satisfy (2.7) and we will be able to apply the Avalanche Principle to them with $\mu = \exp(l\gamma/2)$. The conclusion then follows from the fact that

$$\begin{aligned} \log \|A_m^u(z) \dots A_1^u(z)\| + \sum_{j=2}^{n-1} \log \|A_j^u(z)\| - \sum_{j=1}^{n-1} \log \|A_{j+1}^u(z) A_j^u(z)\| \\ = \log \|A_m^a(z) \dots A_1^a(z)\| + \sum_{j=2}^{n-1} \log \|A_j^a(z)\| - \sum_{j=1}^{n-1} \log \|A_{j+1}^a(z) A_j^a(z)\|. \end{aligned}$$

This identity is a simple consequence of (2.3).

Now we just need to check that the matrices A_j^u satisfy (2.8) and (2.9) with $\mu = \exp(l\gamma/2)$. We have

$$\begin{aligned} \log \|A_j^u(z)\| &\geq \log \left| f_{l_j}^u(z + s_j \omega, \omega, E) \right| \\ &= -\frac{1}{2} \left(\tilde{S}_{l_j}(z + s_j \omega, \omega) + S_{l_j}(z + (s_j + 1)\omega, \omega) \right) + \log \left| f_{l_j}^a(z + s_j \omega, \omega, E) \right| \\ &\geq -Dl_j - (\log l_j)^C + L^a l_j - H(\log l_j)^{C_0} = l_j L - (\log l_j)^C - H(\log l_j)^{C_0} \\ &\geq l \frac{\gamma}{2} \geq \log m. \end{aligned}$$

For the identities we used (2.3) and (2.4). For the second inequality we used Lemma 2.5. The second to last inequality holds for large enough l due to our assumptions. We also have

$$\begin{aligned}
& \log \|A_j^u(z)\| + \log \|A_{j+1}^u(z)\| - \log \|A_{j+1}^u(z)A_j^u(z)\| \\
&= \log \|A_j^a(z)\| + \log \|A_{j+1}^a(z)\| - \log \|A_{j+1}^a(z)A_j^a(z)\| \\
&\leq \log \|M_{l_j}^a(z + s_j\omega)\| + \log \|M_{l_{j+1}}^a(z + s_{j+1}\omega)\| - \log \|f_{l_j+l_{j+1}}^a(z + s_j\omega)\| \\
&\leq l_j L^a + (\log l_j)^C + l_{j+1} L^a + (\log l_{j+1})^C - (l_j + l_{j+1}) L^a + H(\log(l_j + l_{j+1}))^{C_0} \\
&\leq 2(\log(3l))^C + H(\log(6l))^{C_0} \leq \frac{l\gamma}{4} = \frac{1}{2}\log\mu,
\end{aligned}$$

provided l is large enough. Note that we used (2.3) and Proposition 2.2. This concludes the proof. \square

The large deviations estimate for the determinants and the uniform upper bound allows one to use Cartan's estimate. We recall this estimate in the formulation from [GS11].

Definition 2.8. ([GS11, Definition 2.1]) Let $H \gg 1$. For an arbitrary set $\mathcal{B} \subset \mathcal{D}(z_0, 1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \text{Car}_1(H, K)$ if $\mathcal{B} \subset \cup_{j=1}^{j_0} \mathcal{D}(z_j, r_j)$ with $j_0 \leq K$, and $\sum_j r_j \leq \exp(-H)$. If d is a positive integer greater than one and $\mathcal{B} \subset \mathcal{P}(z^0, 1) \subset \mathbb{C}^d$ then we define inductively that $\mathcal{B} \in \text{Car}_d(H, K)$ if, for any $1 \leq j \leq d$, there exists $\mathcal{B}_j \subset \mathcal{D}(z_j^0, 1) \subset \mathbb{C}$, $\mathcal{B}_j \in \text{Car}_1(H, K)$ so that $\mathcal{B}_z^{(j)} := \{(z_1, \dots, z_d) \in \mathcal{B} : z_j = z\} \in \text{Car}_{d-1}(H, K)$ for any $z \in \mathbb{C} \setminus \mathcal{B}_j$.

Lemma 2.9. ([GS11, Lemma 2.4]) Let $\phi(z_1, \dots, z_d)$ be an analytic function defined in a polydisk $\mathcal{P} = \mathcal{P}(z^0, 1)$, $z^0 \in \mathbb{C}^d$. Let $M \geq \sup_{z \in \mathcal{P}} \log |\phi(z)|$, $m \leq \log |\phi(z^0)|$. Given $H \gg 1$, there exists a set $\mathcal{B} \subset \mathcal{P}$, $\mathcal{B} \in \text{Car}_d(H^{1/d}, K)$, $K = C_d H(M - m)$, such that

$$\log |\phi(z)| > M - C_d H(M - m),$$

for any $z \in \mathcal{P}(z^0, 1/6) \setminus \mathcal{B}$.

The following result is a good illustration for the use of Cartan's estimate. It essentially tells us that the large deviations estimate for $f_N^a(x, \omega, E)$ can only fail if E is close to the spectrum of $H^{(N)}(x, \omega)$.

Proposition 2.10. Let $H \gg 1$ and $(\omega, E) \in \mathbb{T}_{c, \alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$. There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$ such that for all $N \geq N_0$ and $x \in \mathbb{T}$, if

$$\log |f_N^a(x, \omega, E)| \leq N L^a(\omega, E) - H(\log N)^{C_0}, \quad (2.10)$$

then $f_N^a(z, \omega, E) = 0$ for some $|z - x| \lesssim N^{-1} \exp(-H)$. Furthermore, there exists a constant $C_1 = C_1(\|a\|_\infty, \|b\|_\infty)$ such that

$$\text{dist}(E, \text{spec}(H^{(N)}(x, \omega))) \lesssim C_1 N^{-1} \exp(-H).$$

Proof. Let $\phi(\zeta) = f_N^a(x + N^{-1}\zeta, \omega, E)$. By the large deviations estimate for determinants (Proposition 2.1) it follows that for large enough N there exists ζ_0 , $|\zeta_0| < 1/100$, such that $|\phi(\zeta_0)| > NL^a(\omega, E) - (\log N)^C$. Using Corollary 2.3 we can apply Cartan's estimate, Lemma 2.9, to ϕ on $\mathcal{D}(\zeta_0, 1)$, to get that $\log|\phi(\zeta)| > NL^a(\omega, E) - H(\log N)^{C_0}$, for $\zeta \in \mathcal{D}(\zeta_0, 1/6) \setminus (\cup_j \mathcal{D}(\zeta_j, r_j))$, with $\sum_j r_j \leq \exp(-H)$. By our assumption (2.10), it follows that $0 \in \mathcal{D}(\zeta_j, r_j)$ for some j . Furthermore there must exist $\zeta' \in \mathcal{D}(\zeta_0, 1/6) \cap \mathcal{D}(\zeta_j, r_j)$ such that $\phi(\zeta') = 0$, otherwise we can use the minimum modulus principle to contradict (2.10). Now, the first claim holds with $z = x + N^{-1}\zeta'$. The last claim follows from the fact that there exists a constant $C_1 = C_1(\|a\|_\infty, \|b\|_\infty)$ such that

$$\|H^{(N)}(z, \omega) - H^{(N)}(x, \omega)\| \leq C_1|z - x|,$$

and the fact that $H^{(N)}(x, \omega)$ is Hermitian. \square

Next we present the key tools for obtaining localization. They are the Poisson formula in terms of Green's function and a bound on the off-diagonal terms of Green's function in terms of the deviations estimate for the determinant f_N^a . We will denote Green's function by $G_N(z, \omega, E) := (H^{(N)}(z, \omega) - E)^{-1}$, or in general $G_\Lambda(z, \omega, E) := (H_\Lambda(z, \omega) - E)^{-1}$. It is known that any solution ψ of the difference equation $H(z, \omega)\psi = E\psi$ satisfies the Poisson formula:

$$\psi(m) = G_{[a,b]}(z, \omega, E)(m, a)\psi(a-1) + G_{[a,b]}(z, \omega, E)(m, b)\psi(b+1), \quad (2.11)$$

for any $[a, b]$ and $m \in [a, b]$. Using Cramer's rule one can explicitly write the entries of Green's function. Namely, we have that $G_N(z, \omega, E)(j, k)$ is given by

$$\begin{cases} \frac{f_{j-1}^a(z, \omega, E)b(z+j\omega)\dots b(z+(k-1)\omega)f_{N-(k+1)}^a(z+(k+1)\omega, \omega, E)}{f_N^a(z, \omega, E)}, & j < k \\ \frac{f_{k-1}^a(z, \omega, E)\tilde{b}(z+k\omega)\dots \tilde{b}(z+(j-1)\omega)f_{N-(j+1)}^a(z+(j+1)\omega, \omega, E)}{f_N^a(z, \omega, E)}, & j > k \\ \frac{f_{j-1}^a(z, \omega, E)f_{N-(k+1)}^a(z+(k+1)\omega, \omega, E)}{f_N^a(z, \omega, E)}, & j = k. \end{cases}$$

Lemma 2.11. *Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E) > \gamma > 0$. There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$, such that for $N \geq N_0$ we have that if*

$$\log|f_N^a(x, \omega, E)| \geq NL_N^a(\omega, E) - K/2,$$

for some $x \in \mathbb{T}$ and $K > (\log N)^{C_0}$, then

$$|G_N(x, \omega, E)(j, k)| \leq \exp(-\gamma|k-j| + K).$$

Proof. Assume $j < k$. Then we have

$$\begin{aligned} |G_N(z, \omega, E)| &= \frac{|f_{j-1}^a(x, \omega, E)| \exp(S_{k-j}(x+j\omega, \omega)) |f_{N-(k+1)}^a(x+(k+1)\omega, \omega, E)|}{|f_N^a(x, \omega, E)|} \\ &\leq \exp\left((j-1)L^a + (k-j)D + (N-k-1)L^a - NL^a + \frac{K}{2} + (\log N)^C\right) \\ &= \exp\left((k-j)(D - L^a) - 2L^a + \frac{K}{2} + (\log N)^C\right) \leq \exp(-\gamma(k-j) + K). \end{aligned}$$

We used Proposition 2.2, Lemma 2.5, and (2.4). The cases $j = k$ and $j > k$ are analogous. \square

Finally, the following result is needed for the Weierstrass Preparation of the determinants (see Proposition 5.2). The statement of the result is adapted to our setting.

Proposition 2.12. ([BV12, Theorem 4.13]) *Let $(\omega, E_0) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E_0) > \gamma > 0$. There exist constants $C_0 = C_0(\alpha)$, $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$, and $N_0 = N_0(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$ such that for any $x_0 \in \mathbb{T}$ and $N \geq N_0$ one has*

$$\#\{E \in \mathbb{R} : f_N^a(x_0, \omega, E) = 0, |E - E_0| < N^{-C_1}\} \leq C_1(\log N)^{C_0}$$

and

$$\#\{z \in \mathbb{C} : f_N^a(z, \omega, E_0) = 0, |z - x_0| < N^{-1}\} \leq C_1(\log N)^{C_0}.$$

3 Localization

In this section we will show that elimination of resonances implies localization. More precisely we will assume that we have the following elimination of resonances result.

Elimination Assumption 3.1. *Let $A = A(\alpha)$ be a fixed constant, much larger than the C_0 constants from Corollary 2.3, Corollary 2.4, and Lemma 2.11. Let $l = 2 \lceil (\log N)^A \rceil$. We assume that there exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0)$ such that for any $N \geq N_0$ there exist constants $\sigma_N \gg \exp(-l^{1/4})$, $Q_N \gg l^3$, and a set $\Omega_N \subset \mathbb{T}$, with the property that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ there exists a set $\mathcal{E}_{N,\omega} \subset \mathbb{R}$ such that for any $x \in \mathbb{T}$ and any integer m , $Q_N \leq |m| \leq N$, we have*

$$\text{dist}(\mathcal{E}^0 \cap \text{spec}(H^{(l_1)}(x, \omega)) \setminus \mathcal{E}_{N,\omega}, \text{spec}(H^{(l_2)}(x + m\omega, \omega))) \geq \sigma_N, \quad (3.1)$$

$$l_1, l_2 \in \{l, l+1, 2l, 2l+1\}.$$

Similarly to [GS11], we could have assumed that we have elimination between any scales l_1, l_2 , $l \leq l_1, l_2 \leq 3l$. However, this would lead to an extra $\log N$ power in our final separation result. We note that for localization it is enough to assume $l_1, l_2 \in \{l, 2l\}$, and that the stronger assumption is needed in the next section, for obtaining separation.

In this section and the next, all the results hold under the implicit assumption that N is large enough, as needed. The lower bound on N will depend on all the parameters of the problem (as in the Elimination Assumption 3.1).

The following lemma is the basic mechanism through which elimination of resonances enters the proof of localization. As a consequence of Proposition 2.10, it shows that the large deviations estimate for $f_l^a(x + m\omega, \omega, E)$ can only fail for shifts m in a “small” interval (that will end up being the localization window).

Lemma 3.2. *For all $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,a} \setminus \Omega_N$, and $E \in \mathcal{E}^0$, $\text{dist}(E, \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^c) \gtrsim \exp(-l^{1/4})$, if we have*

$$\log |f_l^a(x + n_1\omega, \omega, E)| \leq lL_l^a - \sqrt{l}, \quad (3.2)$$

for some $n_1 \in [0, N-1]$, then

$$\log |f_{l'}^a(x + n\omega, \omega, E)| > l' L^a(\omega, E) - \sqrt{l'}, l' \in \{l, l+1, 2l, 2l+1\}, \quad (3.3)$$

for all $n \in [0, N-1] \setminus [n_1 - Q_N, n_1 + Q_N]$.

Proof. Fix $x \in \mathbb{T}$, $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, and $E \in \mathcal{E}^0$, such that

$$\text{dist}\left(E, \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^C\right) \gtrsim \exp(-l^{1/4}). \quad (3.4)$$

Suppose there exists $n_1 \in [0, N-1]$ such that (3.2) holds. By Proposition 2.10 we have that there exists $E_k^{(l)}(x + n_1\omega, \omega)$ such that $\left|E_k^{(l)}(x + n_1\omega, \omega) - E\right| \leq \exp(-l^{1/3})$. Due to (3.4) we have that $E_k^{(l)}(x + n_1\omega, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N,\omega}$. If (3.3) doesn't hold for $n \in [0, N-1] \setminus [n_1 - Q_N, n_1 + Q_N]$, then there exists $E_{k'}^{(l')}(x + n\omega, \omega)$ such that $\left|E_{k'}^{(l')}(x + n\omega, \omega) - E\right| \leq \exp(-l^{1/3})$, and hence

$$\left|E_k^{(l)}(x + n_1\omega, \omega) - E_{k'}^{(l')}(x + n\omega, \omega)\right| \lesssim \exp(-l^{1/3}).$$

This contradicts (3.1), and thus concludes the proof. \square

We can now apply the Avalanche Principle to obtain large deviations estimates at scales larger than l .

Corollary 3.3. *Under the same assumptions as in Lemma 3.2 and with n_1 as in Lemma 3.2, we have*

$$\left|f_{[0,n-1]}^a(x, \omega, E)\right| > \exp(nL^a(\omega, E) - l^3) \quad (3.5)$$

for each $n = kl, kl+1$, $0 \leq n \leq n_1 - Q_N$, and

$$\left|f_{[n,N-1]}^a(x, \omega, E)\right| > \exp((N-n)L^a(\omega, E) - l^3), \quad (3.6)$$

for each $n_1 + Q_N \leq n = N - kl \leq N - 1$, $k \in \mathbb{Z}$.

Proof. We only prove (3.5) for $n = kl$. The other claims follow in the same way.

Suppose that $n = kl$ and (3.5) fails. Then by Proposition 2.10 we have $f_{[0,n-1]}^a(z) = f_n^a(z) = 0$ for z such that $|z - x| \lesssim n^{-1} \exp(-l^3/(\log n)^{C_0}) \lesssim \exp(-l^2)$ (the last inequality holds due to our choice of l in the Elimination Assumption 3.1). Using Corollary 2.4 we can conclude that

$$\log |f_{l'}^a(z + k\omega)| > l' L^a - 2\sqrt{l'}, l' \in \{l, l+1, 2l, 2l+1\},$$

for all $k \in [0, N-1] \setminus [n_1 - Q_N, n_1 + Q_N]$. We can now use Corollary 2.7 and Corollary

2.3 to get

$$\begin{aligned}
\log|f_n^a(z)| &\gtrsim -k \exp\left(-\frac{\gamma}{2}l\right) - \sum_{j=2}^{k-1} \log\|A_j^a(z)\| + \sum_{j=1}^{k-1} \log\|A_{j+1}^a(z)A_j^a(z)\| \\
&\gtrsim -k \exp\left(-\frac{\gamma}{2}l\right) - \sum_{j=2}^{k-1} \log\|M_l^a(z+(j-1)l\omega)\| + \sum_{j=1}^{k-1} \log|f_{2l}^a(z+(j-1)l\omega)| \\
&\gtrsim -k \exp\left(-\frac{\gamma}{2}l\right) - (k-2)\left(lL^a + (\log l)^C\right) + (k-1)\left(2lL^a - 2\sqrt{2}l\right) \\
&\gtrsim k l L^a - 4k\sqrt{l}.
\end{aligned}$$

This contradicts $f_n^a(z) = 0$. Hence we proved that (3.5) holds. \square

We have all we need to obtain localization.

Theorem 3.4. *For all $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,a} \setminus \Omega_N$, if the eigenvalue $E_j^{(N)}(x, \omega)$ is such that $\text{dist}\left(E_j^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^C\right) \gtrsim \exp(-l^{1/4})$, then there exists $\nu_j^{(N)}(x, \omega) \in [0, N-1]$ so that for any $\Lambda = [a, b]$,*

$$\left[\nu_j^{(N)}(x, \omega) - 3Q_N, \nu_j^{(N)}(x, \omega) + 3Q_N\right] \cap [0, N-1] \subset \Lambda \subset [0, N-1],$$

if we let $Q = \text{dist}\left([0, N-1] \setminus \Lambda, \nu_j^{(N)}(x, \omega)\right)$ we have:

1.

$$\sum_{k \in [0, N-1] \setminus \Lambda} \left| \psi_j^{(N)}(x, \omega; k) \right|^2 < \exp(-\gamma Q), \quad (3.7)$$

2.

$$\text{dist}\left(E_j^{(N)}(x, \omega), \text{spec}(H_\Lambda(x, \omega))\right) \lesssim \exp(-\gamma Q). \quad (3.8)$$

Proof. Fix $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,a} \setminus \Omega_N$, and $E = E_j^{(N)}(x, \omega)$, satisfying our assumptions. Let $\nu_j^{(N)}(x, \omega)$ be such that

$$\left| \psi_j^{(N)}\left(x, \omega; \nu_j^{(N)}(x, \omega)\right) \right| = \max_{0 \leq n \leq N-1} \left| \psi_j^{(N)}(x, \omega; n) \right|.$$

Let $\Lambda_0 = [a_0, b_0] \subset [0, N-1]$ be the interval of length l such that

$$\Lambda_0 \supset \left(\left[\nu_j^{(N)}(x, \omega) - l/2, \nu_j^{(N)}(x, \omega) + l/2 \right] \cap [0, N-1] \right).$$

We claim that

$$\log|f_{\Lambda_0}^a(x, \omega, E)| \leq lL^a - \sqrt{l}. \quad (3.9)$$

Otherwise, Lemma 2.11 implies that

$$|G_{\Lambda_0}(x, \omega, E)(j, k)| \leq \exp\left(-\gamma|k-j| + 2\sqrt{l}\right),$$

for all $j, k \in \Lambda_0$. This, together with Poisson's formula (2.11) would contradict the maximality of $\left| \psi_j^{(N)}(x, \omega; \nu_j^{(N)}(x, \omega)) \right|$.

We note for future reference that (3.9) and Proposition 2.10 imply the existence of $E_k^{(l)}(x + a_0\omega, \omega)$ such that

$$\left| E_k^{(l)}(x + a_0\omega, \omega) - E_j^{(N)}(x, \omega) \right| \leq \exp(-l^{1/3}). \quad (3.10)$$

Let $k \in [0, N-1]$, $k \leq \nu_j^{(N)}(x, \omega) - Q$. Due to (3.9) we can apply Corollary 3.3, with $n_1 = a_0$, $n = l[(n_1 - Q_n)/l]$ to get that

$$\log |f_{[0, n-1]}^a(x)| \geq nL^a - l^3.$$

Now we can apply Lemma 2.11 and (2.11) to get

$$\begin{aligned} \left| \psi_j^{(N)}(x, \omega; k) \right|^2 &\leq \left| G_{[0, n-1]}(x, \omega)(k, n-1) \right|^2 \leq \exp(-2\gamma(n-1-k) + 4l^3) \\ &\leq \exp\left(-2\gamma\left(n_1 - Q_N - l - 1 - \nu_j^{(N)}(x, \omega) + Q\right) + 4l^3\right) \leq \exp\left(-\frac{3\gamma Q}{2}\right) \end{aligned}$$

(we used $\nu_j^{(N)}(x, \omega) - n_1 \leq l/2$, $Q \geq 3Q_N \gg l^3$). Similarly, we obtain the same bound when $k \geq \nu_j^{(N)}(x, \omega) + Q$. Summing up these bounds gives us (3.7).

Due to (3.7) we have

$$\left\| \left(H_\Lambda(x, \omega) - E_j^{(N)}(x, \omega) \right) \left(\psi_j^{(N)}|_\Lambda \right) \right\| < \exp(-\gamma Q).$$

Since H_Λ is Hermitian, and $\left\| \psi_j^{(N)}|_\Lambda \right\| > 1 - \exp(-\gamma Q)$, we can conclude that

$$\text{dist}\left(E_j^{(N)}(x, \omega), \text{spec}(H_\Lambda(x, \omega))\right) < \exp(-\gamma Q)(1 - \exp(-\gamma Q))^{-1} \lesssim \exp(-\gamma Q).$$

□

4 Separation of Eigenvalues

In this section we continue to work under the Elimination Assumption 3.1. The basic idea behind proving separation of eigenvalues is to use the fact that the eigenvectors are orthogonal, and so they cannot be too close. It is known that if E is an eigenvalue of the Dirichlet problem on $[0, N-1]$ then $\mathfrak{f} := \left(f_{[0, n-1]}^a(x, \omega, E) \right)_{n=0}^{N-1}$ is an eigenvector associated with E ($f_{[0, -1]}^a = 1$). Note that we are assuming the boundary conditions $\mathfrak{f}(-1) = \mathfrak{f}(N) = 0$. We will need the following lemma to argue that if two localized eigenvalues are close enough, then they have eigenvectors which are also close, at least before the localization window.

Lemma 4.1. *Let $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$, and suppose that*

$$\text{dist}\left(E_j^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^C\right) \gtrsim \exp(-l^{1/4})$$

for some j . If E is such that $|E - E_j^{(N)}(x, \omega)| \leq N^{-C_1}$, with C_1 as in Corollary 2.4, then

$$\begin{aligned} & \left| f_{[0,n-1]}^a(x, \omega, E) - f_{[0,n-1]}^a\left(x, \omega, E_j^{(N)}(x, \omega)\right) \right| \\ & \leq \exp(2l^3) \left| E - E_j^{(N)}(x, \omega) \right| \left| f_{[0,n-1]}^a\left(x, \omega, E_j^{(N)}(x, \omega)\right) \right|, \end{aligned}$$

for each $n = kl, kl+1$, $k \in \mathbb{Z}$, $0 \leq n \leq \nu_j^{(N)}(x, \omega) - 2Q_N$, where $\nu_j^{(N)}(x, \omega)$ is the localization center corresponding to $E_j^{(N)}(x, \omega)$ (as in Theorem 3.4).

Proof. This follows immediately from (2.5) and Corollary 3.3. \square

The next lemma shows that if two localized eigenvalues are close enough, then their localization centers are also close.

Lemma 4.2. *Let $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ and suppose that*

$$\text{dist}\left(E_{j_i}^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^C\right) \gtrsim \exp(-l^{1/4}), i = 1, 2.$$

If $|E_{j_1}^{(N)}(x, \omega) - E_{j_2}^{(N)}(x, \omega)| \leq \sigma_N/2$, then both eigenvalues are localized and if we denote their localization centers by $\nu_{j_i}^{(N)}(x, \omega)$, $i = 1, 2$, we have $|\nu_{j_1}^{(N)}(x, \omega) - \nu_{j_2}^{(N)}(x, \omega)| < 2Q_N$.

Proof. As was noted in the proof of Theorem 3.4 (see (3.10)) we have that

$$\left| E_{j_i}^{(N)}(x, \omega) - E_{k_i}^{(l)}(x + n_i\omega, \omega) \right| \leq \exp(-l^{1/3}), i = 1, 2, \quad (4.1)$$

where n_i are such that $|\nu_{j_i}^{(N)}(x, \omega) - n_i| \leq l/2$, $i = 1, 2$.

Suppose that $|n_1 - n_2| \geq Q_N$. Due to (4.1) we have that $E_{k_1}^{(l)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N,\omega}$ and hence, by (3.1) we have

$$\left| E_{k_1}^{(l)}(x + n_1\omega, \omega) - E_{k_2}^{(l)}(x + n_2\omega, \omega) \right| \geq \sigma_N.$$

The above inequality together with (4.1) and the assumption that $\sigma_N \gg \exp(-l^{1/4})$, implies that $|E_{j_1}^{(N)}(x, \omega) - E_{j_2}^{(N)}(x, \omega)| > \sigma_N/2$, contradicting our assumptions. So, we must have $|n_1 - n_2| < Q_N$ and consequently $|\nu_{j_1}^{(N)}(x, \omega) - \nu_{j_2}^{(N)}(x, \omega)| \leq Q_N + l < 2Q_N$. \square

We are now ready to prove a first version of separation, based on the size of the localization window. This is a generalization of [GS11, Proposition 7.1].

Proposition 4.3. *There exists a constant $C_0 = C_0(\|a\|_\infty, \|b\|_\infty, E^0)$ such that for all $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$, if $\text{dist}\left(E_j^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^c\right) \gtrsim \exp(-l^{1/4})$ for some j , then*

$$\left| E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega) \right| > \exp(-C_0 Q_N)$$

for any $k \neq j$.

Proof. Fix $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ and $E_1 = E_j^{(N)}(x, \omega)$ satisfying the assumptions. Suppose there exists $E_2 = E_k^{(N)}(x, \omega) \neq E_1$ such that $|E_1 - E_2| \leq \exp(-C_0 Q_N)$. This implies that $\text{dist}\left(E_k^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^c\right) \gtrsim \exp(-l^{1/4})$ (recall that $Q_N \gg l^3$). Hence, by Theorem 3.4, both $E_j^{(N)}(x, \omega)$, and $E_k^{(N)}(x, \omega)$ are localized.

We know $\mathbf{f}_i := \left(f_{[0,n-1]}^a(x, \omega, E_i) \right)_{n=0}^{N-1}$ are eigenvectors corresponding to E_i , $i = 1, 2$. Furthermore $\mathbf{f}_i(-1) = \mathbf{f}_i(N) = 0$. If we let

$$\Lambda = [a, b] = [0, N-1] \cap \left[\nu_j^{(N)}(x, \omega) - 5Q_N, \nu_j^{(N)}(x, \omega) + 5Q_N \right],$$

then due to (3.7) and Lemma 4.2 we have

$$\sum_{n \in [0, N-1] \setminus \Lambda} |\mathbf{f}_i(n)|^2 \lesssim \exp(-5\gamma Q_N) \sum_{n \in \Lambda} |\mathbf{f}_i(n)|^2, i = 1, 2, \quad (4.2)$$

and consequently

$$\sum_{n \in [0, N-1] \setminus \Lambda} |\mathbf{f}_1(n) - \mathbf{f}_2(n)|^2 \lesssim \exp(-5\gamma Q_N) \sum_{n \in \Lambda} (|\mathbf{f}_1(n)|^2 + |\mathbf{f}_2(n)|^2). \quad (4.3)$$

Let

$$m = \begin{cases} [(a-2)/l]l & , a > l+1 \\ -1 & , a \leq l+1 \end{cases}.$$

For $n \in \Lambda$ we have

$$\begin{aligned} |\mathbf{f}_1(n) - \mathbf{f}_2(n)|^2 &\leq \left\| \begin{pmatrix} \mathbf{f}_1(n+1) \\ \mathbf{f}_1(n) \end{pmatrix} - \begin{pmatrix} \mathbf{f}_2(n+1) \\ \mathbf{f}_2(n) \end{pmatrix} \right\|^2 \\ &= \left\| M_{[m+1,n]}^a(E_1) \begin{pmatrix} \mathbf{f}_1(m+1) \\ \mathbf{f}_1(m) \end{pmatrix} - M_{[m+1,n]}^a(E_2) \begin{pmatrix} \mathbf{f}_2(m+1) \\ \mathbf{f}_2(m) \end{pmatrix} \right\|^2 \\ &\lesssim \left\| (M_{[m+1,n]}^a(E_1) - M_{[m+1,n]}^a(E_2)) \begin{pmatrix} \mathbf{f}_1(m+1) \\ \mathbf{f}_1(m) \end{pmatrix} \right\|^2 \\ &\quad + \left\| M_{[m+1,n]}^a(E_2) \begin{pmatrix} \mathbf{f}_1(m+1) - \mathbf{f}_2(m+1) \\ \mathbf{f}_1(m) - \mathbf{f}_2(m) \end{pmatrix} \right\|^2 \\ &\lesssim \exp(CQ_N) |E_1 - E_2|^2 (|\mathbf{f}_1(m+1)|^2 + |\mathbf{f}_1(m)|^2) \\ &\quad + \exp(CQ_N) (|\mathbf{f}_1(m+1) - \mathbf{f}_2(m+1)|^2 + |\mathbf{f}_1(m) - \mathbf{f}_2(m)|^2) \\ &\lesssim \exp(CQ_N) |E_1 - E_2|^2 (|\mathbf{f}_1(m+1)|^2 + |\mathbf{f}_1(m)|^2). \quad (4.4) \end{aligned}$$

For the second to last inequality we used Corollary 2.4, Proposition 2.2, and the fact that $n - m \lesssim Q_N$ for $n \in \Lambda$. For the last inequality, in the case when $a > l + 1$, we used Lemma 4.1 and the assumption that $Q_N \gg l^3$. When $a \leq l + 1$ the last inequality holds trivially since $f_i(-1) = 0$, $f_i(0) = 1$, $i = 1, 2$.

Assume that $a > 0$. We have that either $m, m + 1 \in [0, N - 1] \setminus \Lambda$, or $m = -1$, $m + 1 \in [0, N - 1] \setminus \Lambda$. Since $f_1(-1) = 0$, using (4.2) and (4.4) we can conclude in either case that

$$\sum_{n \in \Lambda} |f_1(n) - f_2(n)|^2 \lesssim \exp(-CQ_N) \sum_{n \in \Lambda} |f_1(n)|^2. \quad (4.5)$$

If $a = 0$, then this follows trivially from (4.4). From (4.5), (4.3), and the fact that f_1 and f_2 are orthogonal, we get that

$$\|f_1 - f_2\|^2 = \sum_{n \in [0, N-1]} (|f_1(n)|^2 + |f_2(n)|^2) \lesssim \exp(-CQ_N) \sum_{n \in \Lambda} (|f_1(n)|^2 + |f_2(n)|^2).$$

This is absurd, so we cannot have $|E_1 - E_2| \leq \exp(-C_0Q_N)$. \square

Next we use a bootstrapping argument to improve the separation from the previous proposition.

Theorem 4.4. *Suppose there exists N' , $2Q_N^2 \leq N' < N$, such that $\exp(-C_0Q_{N'}) \geq \sigma_N$, with C_0 as in the previous proposition. Then for all $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus (\Omega_N \cup \Omega_{N'})$, if $\text{dist}\left(E_j^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup \mathcal{E}_{N',\omega} \cup (\mathcal{E}^0)^C\right) \geq \sigma_N$ for some j , then*

$$\left|E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega)\right| > \sigma_N/2$$

for any $k \neq j$.

Proof. Fix $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus (\Omega_N \cup \Omega_{N'})$, and j , such that $E_1 = E_j^{(N)}(x, \omega)$ satisfies the assumptions. Suppose that there exists $E_2 = E_k^{(N)}(x, \omega) \neq E_1$ such that $|E_1 - E_2| \leq \sigma_N/2$. We have that $\text{dist}\left(E_k^{(N)}(x, \omega), \mathcal{E}_{N,\omega} \cup (\mathcal{E}^0)^C\right) \geq \sigma_N/2 \gg \exp(-l^{1/4})$, and due to Lemma 4.2 it is possible to choose an interval $\Lambda \subset [0, N - 1]$ of length N' such that $\Lambda \supset \left[\nu_i^{(N)}(x, \omega) - Q_N^2, \nu_i^{(N)}(x, \omega) + Q_N^2\right]$, $i \in \{j, k\}$. By (3.8) we know that there exist $E'_1, E'_2 \in \text{spec}(H_\Lambda(x, \omega))$ such that $|E_i - E'_i| \lesssim \exp(-\gamma Q_N^2)$. Note that $E'_1 \neq E'_2$, since otherwise $|E_1 - E_2| \lesssim \exp(-\gamma Q_N^2)$, contradicting the conclusion of Proposition 4.3. We also have that

$$\text{dist}\left(E'_1, \mathcal{E}_{N',\omega} \cup (\mathcal{E}^0)^C\right) \gtrsim \sigma_N - \exp(-\sigma Q_N^2) \geq \sigma_N/2 \gg \exp(-l^{1/4}) \geq \exp(-l'^{1/4}),$$

where $l' = 2 \left\lceil (\log N')^A \right\rceil$, with A as in the Elimination Assumption 3.1. Applying Proposition 4.3 at scale N' we get that $|E'_1 - E'_2| > \exp(-C_0Q_{N'}) \geq \sigma_N$, and consequently $|E_1 - E_2| > \sigma_N - \exp(-Q_N^2) \geq \sigma_N/2$. We arrived at a contradiction, and the proof is concluded. \square

5 Elimination, Localization, and Separation via Resultants

In this section we will first obtain the elimination of resonances via resultants using the abstract results from [GS11]. Then we will apply the abstract results of the previous two sections to get concrete localization and separation.

As was mentioned in the introduction, we first need to apply the Weierstrass Preparation Theorem to the determinants. For convenience we recall a version of the Weierstrass Preparation Theorem. In what follows $f(z, w)$ is a function defined on the polydisk $\mathcal{P} = \mathcal{D}(z_0, R_0) \times \mathcal{P}(w^0, R_0)$, $z_0 \in \mathbb{C}$, $w^0 \in \mathbb{C}^d$, $1/2 \geq R_0 > 0$.

Lemma 5.1. ([GS11, Proposition 2.26]) *Assume that $f(\cdot, w)$ has no zeros on some circle $|z - z_0| = r_0$, $0 < r_0 < R_0/2$, for any $w \in \mathcal{P}_1 = \mathcal{P}(w^0, r_1)$ where $0 < r_1 < R_0$. Then there exist a polynomial $P(z, w) = z^k + a_{k-1}(w)z^{k-1} + \dots + a_0(w)$ with $a_j(w)$ analytic in \mathcal{P}_1 and an analytic function $g(z, w)$, $(z, w) \in \mathcal{D}(z_0, r_0) \times \mathcal{P}_1$ so that the following statements hold:*

1. $f(z, w) = P(z, w)g(z, w)$ for any $(z, w) \in \mathcal{D}(z_0, r_0) \times \mathcal{P}_1$,
2. $g(z, w) \neq 0$ for any $(z, w) \in \mathcal{D}(z_0, r_0) \times \mathcal{P}_1$,
3. For any $w \in \mathcal{P}_1$, $P(\cdot, w)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, r_0)$.

We can now obtain the Weierstrass Preparation of the determinants.

Proposition 5.2. *Given $x_0 \in \mathbb{T}$, $(\omega_0, E_0) \in \mathbb{T}_{c, \alpha} \times \mathbb{C}$ such that $L(\omega_0, E_0) > \gamma > 0$, there exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$, $C_0 = C_0(\alpha)$, so that for any $N \geq N_0$ there exist $r_0 \simeq N^{-1}$, a polynomial $P_N(z, \omega, E) = z^k + a_{k-1}(\omega, E)z^{k-1} + \dots + a_0(\omega, E)$, with $a_j(\omega, E)$ analytic in $\mathcal{D}(E_0, r_1) \times \mathcal{D}(\omega_0, r_1)$, $r_1 = \exp\left(-(\log N)^{C_0}\right)$, and an analytic function $g_N(z, \omega, E)$, $(z, \omega, E) \in \mathcal{P} := \mathcal{D}(x_0, r_0) \times \mathcal{D}(E_0, r_1) \times \mathcal{D}(\omega_0, r_1)$ such that:*

1. $f_N^a(z, \omega, E) = P_N(z, \omega, E)g_N(z, \omega, E)$,
2. $g_N(z, \omega, E) \neq 0$ for any $(z, \omega, E) \in \mathcal{P}$,
3. For any $(\omega, E) \in \mathcal{D}(\omega_0, r_1) \times \mathcal{D}(E_0, r_1)$ the polynomial $P_N(\cdot, \omega, E)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, r_0)$,
4. $k = \deg P_N(\cdot, \omega, E) \leq (\log N)^{C_0}$.

Proof. Let $f(\zeta, w_1, w_2) := f_N^a(x_0 + N^{-1}\zeta, \omega_0 + N^{-C}w_1, E_0 + N^{-C}w_2)$, where C is larger than the C_1 constants from Corollary 2.3 and Corollary 2.4. By the large deviations estimate for determinants (Proposition 2.1) it follows that (for large enough N) there exists ζ_0 , $|\zeta_0| < 1/100$, such that $|f(\zeta_0, 0, 0)| > NL_N(\omega_0, E_0) - (\log N)^C$. Using Corollary 2.3 we can apply Cartan's estimate (Lemma 2.9) to $\phi(\zeta) = f(\zeta, 0, 0)$ on $\mathcal{D}(\zeta_0, 1)$, to get that there exists $\mathcal{B} \in \text{Car}_1\left(\log N, (\log N)^C\right)$ such that

$$|f(\zeta, 0, 0)| > \exp\left(NL^a(\omega_0, E_0) - (\log N)^C\right), \quad (5.1)$$

for $\zeta \in \mathcal{D}(\zeta_0, 1/6) \setminus \mathcal{B}$. In particular, from Definition 2.8, we can conclude there exists $r \in (1/5, 1/6)$ such that (5.1) holds for $|\zeta| = r$. Using (2.5) we have

$$\begin{aligned} |f(\zeta, w_1, w_2)| &\geq |f(\zeta, 0, 0)| - |f(\zeta, 0, 0) - f(\zeta, w_1, w_2)| \\ &\geq \exp\left(NL^a(\omega_0, E_0) - (\log N)^C\right) \\ &\quad - \exp\left(NL^a(\omega_0, E_0) + (\log N)^C\right) N^{-C}(|w_1| + |w_2|) > 0, \end{aligned}$$

for $|\zeta| = r$, $|w_1|, |w_2| \leq \exp\left(-(\log N)^C\right)$. Now the first three claims follow by applying Lemma 5.1 with $r_0 = rN^{-1}$ and $r_1 = \exp\left(-(\log N)^C\right)$. The last claim is a consequence of Proposition 2.12. \square

Next we recall the abstract version of the elimination via resultants obtained by Goldstein and Schlag. Given $w^0 \in \mathbb{C}^d$, $r = (r_1, \dots, r_d)$, $r_i > 0$, $i = 1, \dots, d$, we let

$$S_{w^0, r}(w) = (r_1^{-1}(w_1 - w_1^0), \dots, r_d^{-1}(w_d - w_d^0)).$$

We will use the notation $\mathcal{Z}(f)$ for the zeros of a function f . We also let $\mathcal{Z}(f, S) := \mathcal{Z}(f) \cap S$ and $\mathcal{Z}(f, r) := \mathcal{Z}(f, \mathbb{H}_r)$.

Lemma 5.3. ([GS11, Lemma 5.4]) *Let $P_s(z, w) = z^{k_s} + a_{s, k_s-1}(w)z^{k_s-1} + \dots + a_{s, 0}(w)$, $z \in \mathbb{C}$, $s = 1, 2$, where $a_{s, j}(w)$ are analytic functions defined on a polydisk $\mathcal{P} = \mathcal{P}(w^0, r)$, $w^0 \in \mathbb{C}^d$. Assume that $k_s > 0$, $s = 1, 2$, and set $k = k_1 k_2$. Suppose that for any $w \in \mathcal{P}$ the zeros of $P_s(\cdot, w)$ belong to the same disk $\mathcal{D}(z_0, r_0)$, $r_0 \ll 1$, $s = 1, 2$. Let $|t| > 16kr_0r^{-1}$. Given $H \gg 1$ there exists a set*

$$\mathcal{B}_{H, t} \subset \tilde{\mathcal{P}} := \mathcal{D}(w_1^0, 8kr_0/|t|) \times \prod_{j=2}^d \mathcal{D}(w_j^0, r/2)$$

such that $S_{w^0, (16kr_0|t|^{-1}, r, \dots, r)}(\mathcal{B}_{H, t}) \in \text{Car}_d(H^{1/d}, K)$, $K = CHk$ and for any $w \in \tilde{\mathcal{P}} \setminus \mathcal{B}_{H, t}$ one has

$$\text{dist}(\mathcal{Z}(P_1(\cdot, w)), \mathcal{Z}(P_2(\cdot + t(w_1 - w_1^0), w))) \geq e^{-CHk}.$$

We can now prove the elimination of resonances via resultants. This is a generalization of [GS11, Proposition 5.5].

Proposition 5.4. *There exist constants $l_0 = l_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0)$, c_0 , $C_0 = C_0(\alpha)$ such that for any $l \geq l' \geq l_0$, t with $|t| \geq \exp\left((\log l)^{C_0}\right)$, and $H \gg 1$, there exists a set $\Omega_{l, l', t, H} \subset \mathbb{T}$, with*

$$\text{mes}(\Omega_{l, l', t, H}) < \exp\left((\log l)^{C_0} - \sqrt{H}\right), \text{compl}(\Omega_{l, l', t, H}) < |t|H \exp\left((\log l)^{C_0}\right),$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_{l, l', t, H}$ there exists a set $\mathcal{E}_{l, l', t, H, \omega}$ with

$$\text{mes}(\mathcal{E}_{l, l', t, H, \omega}) < |t| \exp\left((\log l)^{C_0} - \sqrt{H}\right), \text{compl}(\mathcal{E}_{l, l', t, H, \omega}) < |t|H \exp\left((\log l)^{C_0}\right),$$

such that:

1. For any $E \in \mathcal{E}^0 \setminus \mathcal{E}_{l,l',t,H,\omega}$ we have

$$\text{dist}\left(\mathcal{Z}(f_l^a(\cdot, \omega, E), c_0 l^{-1}), \mathcal{Z}(f_{l'}^a(\cdot + t\omega, \omega, E), c_0 l^{-1})\right) \geq \exp\left(-H(\log l)^{C_0}\right). \quad (5.2)$$

2. For any $x \in \mathbb{T}$ we have

$$\begin{aligned} \text{dist}\left(\mathcal{E}^0 \cap \text{spec}(H^{(l)}(x, \omega)) \setminus \mathcal{E}_{l,l',t,H,\omega}, \text{spec}(H^{(l')}(x + t\omega, \omega))\right) \\ \geq \exp\left(-H(\log l)^{3C_0}\right). \end{aligned} \quad (5.3)$$

Proof. Let $x_0 \in \mathbb{T}$, $E_0 \in \mathcal{E}^0$, and $\omega_0 \in \Omega^0 \cap \mathbb{T}_{c,\alpha}$. Using Proposition 5.2 we can write

$$f_l^a(z, \omega, E) = P_1(z, \omega, E)g_1(z, \omega, E)$$

and

$$f_{l'}^a(z + t\omega_0, \omega, E) = P_2(z, \omega, E)g_2(z, \omega, E),$$

on $\mathcal{P}_0 = \mathcal{D}(x_0, r_0) \times \mathcal{D}(E_0, r_1) \times \mathcal{D}(\omega_0, r_1)$, where $r_0 \simeq l^{-1}$, $r_1 = \exp\left(-(\log l)^C\right)$. The functions g_i , $i = 1, 2$, don't vanish on \mathcal{P}_0 , and the polynomials P_i , $i = 1, 2$ are of degrees k_i , $i = 1, 2$, $k_i \leq (\log l)^C$. Applying Lemma 5.3 to the polynomials $P_1(\cdot, \omega, E)$ and $P_2(\cdot + t(\omega - \omega_0), \omega, E)$, with $|t| \geq \exp\left((\log l)^C\right) > 16k_1k_2r_0r_1^{-1}$, yields that there exists $\mathcal{B}_{H,t} \subset \tilde{\mathcal{P}}_0 := \mathcal{D}(\omega_0, 8kr_0/|t|) \times \mathcal{D}(E_0, r_1/2)$, with

$$\left\{\left(\frac{|t|(\omega - \omega_0)}{16kr_0}, \frac{E - E_0}{r_1}\right) : (\omega, E) \in \mathcal{B}_{H,t}\right\} \in \text{Car}_2\left(H^{1/2}, H(\log l)^C\right), \quad (5.4)$$

so that for any $(\omega, E) \in \tilde{\mathcal{P}}_0 \setminus \mathcal{B}_{H,t}$ we have

$$\text{dist}\left(\mathcal{Z}(P_1(\cdot, \omega, E)), \mathcal{Z}(P_2(\cdot + t(\omega - \omega_0), \omega, E))\right) \geq e^{-H(\log l)^C},$$

which implies that

$$\text{dist}\left(\mathcal{Z}(f_l^a(\cdot, \omega, E), \mathcal{D}(x_0, r_0)), \mathcal{Z}(f_{l'}^a(\cdot + t\omega, \omega, E), \mathcal{D}(x_0, r_0))\right) \geq e^{-H(\log l)^C}. \quad (5.5)$$

Let \mathcal{N}_x be an $r_0/2$ -net covering \mathbb{T} , such that $\{z : |\text{Im} z| < c_0 l^{-1}\} \subset \cup_{x \in \mathcal{N}_x} \mathcal{D}(x, r_0/2)$ (for this c_0 has to be small enough, depending on the absolute constants in $r_0 \simeq l^{-1}$). Let \mathcal{N}_ω be a $8kr_0/|t|$ -net covering $\Omega^0 \cap \mathbb{T}_{c,\alpha}$, \mathcal{N}_E a $r_1/2$ -net covering \mathcal{E}^0 , and $\{(x_j, \omega_j, E_j)\}_j = \mathcal{N}_x \times \mathcal{N}_\omega \times \mathcal{N}_E$. Denote by $\mathcal{B}_{H,t,j}$ the bad set corresponding (as above) to (x_j, ω_j, E_j) . By (5.4) and Definition 2.8 we have that there exists Ω_j , with

$$\text{mes}(\Omega_j) \leq 16kr_0|t|^{-1} \exp\left(-\sqrt{H}\right), \text{compl}(\Omega_j) \leq H(\log l)^C,$$

so that for each $\omega \in \mathcal{D}(\omega_j, 8kr_0/t) \setminus \Omega_j$ we have $(\mathcal{B}_{H,t,j})_\omega^{(1)} =: \mathcal{E}_{j,\omega}$ is such that

$$\text{mes}(\mathcal{E}_{j,\omega}) \leq r_1 \exp\left(-\sqrt{H}\right), \text{compl}(\mathcal{E}_{j,\omega}) \leq H(\log l)^C.$$

We define $\Omega_{l,l',t,H} := \cup_j \Omega_j$ and $\mathcal{E}_{l,l',t,H,\omega} := \cup_j \mathcal{E}_{j,\omega}$, for $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_{l,l',t,H}$. The measure and complexity bounds for these sets are straightforward to check. If (5.2) fails, there would exist $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_{l,l',t,H}$, $E \in \mathcal{E}^0 \setminus \mathcal{E}_{l,l',t,H,\omega}$, and z_1, z_2 , $|\operatorname{Im} z_1|, |\operatorname{Im} z_2| < c_0 l^{-1}$, $|z_1 - z_2| < \exp\left(-H(\log l)^{C_0}\right)$ such that

$$f_l^a(z_1, \omega, E) = f_{l'}^a(z_2 + t\omega, \omega, E) = 0.$$

By our choice of covering nets, we have that $(z_1, \omega, E) \in \mathcal{D}(x_j, r_0/2) \times \mathcal{D}(\omega_j, 8kr_0/|t|) \times \mathcal{D}(E_j, r_1/2)$ for some j . Since $|z_1 - z_2| < \exp\left(-H(\log l)^{C_0}\right)$, we can conclude that we have $(z_i, \omega, E) \in \mathcal{D}(x_j, r_0) \times \mathcal{D}(\omega_j, 8kr_0/|t|) \times \mathcal{D}(E_j, r_1/2)$, which contradicts (5.5). This proves (5.2).

If (5.3) fails, there would exist $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_{l,l',t,H}$, $E_1 \in \mathcal{E}^0 \setminus \mathcal{E}_{l,l',t,H,\omega}$, $E_2 \in \mathbb{C}$, $|E_1 - E_2| < \exp\left(-H(\log l)^{3C_0}\right)$, and $x \in \mathbb{T}$ such that

$$f_l^a(x, \omega, E_1) = f_{l'}^a(x + t\omega, \omega, E_2) = 0.$$

By Corollary 2.4 we have

$$\begin{aligned} |f_{l'}^a(x + t\omega, \omega, E_1)| &= |f_{l'}^a(x + t\omega, \omega, E_1) - f_{l'}^a(x + t\omega, \omega, E_2)| \\ &\leq |E_1 - E_2| \exp\left(l' L^a(\omega, E_1) + (\log l')^C\right) \leq \exp\left(l' L^a(\omega, E_1) - H(\log l)^{2C_0}\right). \end{aligned}$$

By Proposition 2.10, there exists z , $|z - x| \lesssim l'^{-1} \exp\left(-H(\log l)^{C_0}\right)$ such that

$$f_{l'}^a(z + t\omega, \omega, E_1) = 0.$$

This contradicts (5.2), and thus we proved (5.3). \square

Next we state the elimination of resonances as in the Elimination Assumption 3.1.

Corollary 5.5. *Fix $A > 1$. There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, A)$, $C_0 = C_0(\alpha)$, such that for any $N \geq N_0$ there exists a set Ω_N , with*

$$\operatorname{mes}(\Omega_N) < \exp\left(-(\log N)^2\right), \operatorname{compl}(\Omega_N) < N^2 \exp\left((\log \log N)^{C_0}\right),$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ there exists a set $\mathcal{E}_{N,\omega}$, with

$$\operatorname{mes}(\mathcal{E}_{N,\omega}) < \exp\left(-(\log N)^2\right), \operatorname{compl}(\mathcal{E}_{N,\omega}) < N^2 \exp\left((\log \log N)^{C_0}\right),$$

such that for any $x \in \mathbb{T}$ and any integer m , $\exp\left((\log \log N)^{C_0}\right) \leq |m| \leq N$, we have

$$\operatorname{dist}\left(\mathcal{E}^0 \cap \operatorname{spec}\left(H^{(l_1)}(x, \omega) \setminus \mathcal{E}_{N,\omega}\right), \operatorname{spec}\left(H^{(l_2)}(x + m\omega, \omega)\right)\right) \geq \exp\left(-(\log N)^6\right),$$

$l_1, l_2 \in \{l, l+1, 2l, 2l+1\}$, where $l = 2\left[(\log N)^A\right]$.

Proof. It is straightforward to see how this follows from Proposition 5.4 by letting $H = (\log N)^5$. \square

We now have that the Elimination Assumption 3.1 is satisfied with $A = A(\alpha) \gg 1$, $Q_N = \exp((\log \log N)^{C_0})$, $\sigma_N = \exp(-(\log N)^6)$, and $\Omega_N, \mathcal{E}_{N,\omega}$ as in Corollary 5.5. The next result follows immediately from Theorem 3.4.

Proposition 5.6. *There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0)$, $C_0 = C_0(\alpha)$, such that for any $N \geq N_0$ there exists a set Ω_N , with*

$$\text{mes}(\Omega_N) < \exp(-(\log N)^2), \text{compl}(\Omega_N) < N^2 \exp((\log \log N)^{C_0}),$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ there exists a set $\tilde{\mathcal{E}}_{N,\omega}$, with

$$\text{mes}(\tilde{\mathcal{E}}_{N,\omega}) \lesssim \exp(-(\log N)^2), \text{compl}(\tilde{\mathcal{E}}_{N,\omega}) \lesssim N^2 \exp((\log \log N)^{C_0}),$$

such that for any $x \in \mathbb{T}$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \tilde{\mathcal{E}}_{N,\omega}$, for some j , then there exists a point $\nu_j^{(N)}(x, \omega) \in [0, N-1]$ so that for any $\Lambda = [a, b]$,

$$\left[\nu_j^{(N)}(x, \omega) - 3Q_N, \nu_j^{(N)}(x, \omega) + 3Q_N \right] \cap [0, N-1] \subset \Lambda \subset [0, N-1],$$

$Q_N = \exp((\log \log N)^{C_0})$, if we let $Q = \text{dist}([0, N-1] \setminus \Lambda, \nu_j^{(N)}(x, \omega))$ we have:

1.

$$\sum_{k \in [0, N-1] \setminus \Lambda} \left| \psi_j^{(N)}(x, \omega; k) \right|^2 < \exp(-\gamma Q), \quad (5.6)$$

2.

$$\text{dist}\left(E_j^{(N)}(x, \omega), \text{spec}(H_\Lambda(x, \omega))\right) \lesssim \exp(-\gamma Q). \quad (5.7)$$

The next result follows immediately from Proposition 4.3. This is a generalization to the Jacobi case of [GS11, Proposition 7.1].

Proposition 5.7. *Let $\delta \in (0, 1)$ and let $\Omega_N, \tilde{\mathcal{E}}_{N,\omega}$ be as in the previous proposition. There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, \delta)$, such that for $N \geq N_0$, $x \in \mathbb{T}$, $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \tilde{\mathcal{E}}_{N,\omega}$, for some j , then*

$$\left| E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega) \right| > \exp(-N^\delta)$$

for all $k \neq j$.

6 Abstract Elimination of Resonances via Slopes

In this section we will obtain elimination of resonances via slopes (as discussed in the introduction) in an abstract setting. We begin by presenting the assumptions under which we will be working.

Let $e(x) = e^{2\pi i x}$. Let $P(x, y, z)$ be a polynomial of degree at most d_1 for any fixed x , and of degree at most d_2 for any fixed y . Let $f_j : \mathbb{R}^2 \rightarrow \mathbb{R}$, $j = 1, \dots, n$ be functions which are real-analytic and 1-periodic in each variable, and with the property that

$$P(e(x), e(y), f_j(x, y)) = 0, (x, y) \in \mathbb{R}^2, j = 1, \dots, n.$$

Clearly, there exist constants C_0 and C_1 such that

$$|\partial_x f_j(x, y)| \leq C_0, |\partial_y f_j(x, y)| \leq C_1, x, y \in \mathbb{R}, j = 1, \dots, n. \quad (6.1)$$

Equivalently we will have

$$|f_j(x, y) - f_j(x', y')| \leq C_0|x - x'| + C_1|y - y'|, x, x', y, y' \in \mathbb{R}, j = 1, \dots, n. \quad (6.2)$$

Furthermore, we assume that there exist constants c_0, r_0, C_2, C_3 , a set $\mathcal{Y}^0 \subset [0, 1]$, and an interval \mathcal{Z}^0 , such that for every $y \in \mathcal{Y}^0$ there exists a set \mathcal{Z}_y^0 , with

$$\text{mes}(\mathcal{Z}_y^0) \leq c_0, \text{compl}(\mathcal{Z}_y^0) \leq C_2,$$

such that for any $x \in \mathbb{R}$, if $f_j(x, y) \in \mathcal{Z}^0 \setminus \mathcal{Z}_y^0$, for some j , then

$$|\partial_x f_j(x, y) - \partial_x f_j(x, y')| \leq C_3|y - y'|, \quad (6.3)$$

for any $y' \in \mathbb{R}$ such that $|y - y'| \leq r_0$. The rather convoluted form of the assumption is motivated by the concrete estimate that we have for eigenvalues (see Corollary 7.2).

By a Sard-type argument we show that for fixed y , after removing some thin horizontal strips from the graphs of $f_j(\cdot, y)$ we have control over the slopes. Furthermore, these strips are stable under small perturbations in y . We refer to [GS11, Lemma 10.9-10] for similar considerations.

Lemma 6.1. *Fix $\tau > 0$ and let $\delta = \min\{r_0, \tau/C_3, \tau/C_1\}$. For each $y \in \mathcal{Y}^0$ there exists a set \mathcal{Z}_y , with*

$$\text{mes}(\mathcal{Z}_y) \lesssim (n + d_2^2 + C_2)\tau + c_0, \text{compl}(\mathcal{Z}_y) \lesssim d_2^2 + C_2, \quad (6.4)$$

such that for any $x \in \mathbb{R}$ and $y' \in (y - \delta, y + \delta)$, if $f_j(x, y') \in \mathcal{Z}^0 \setminus \mathcal{Z}_y$, for some j , then $|\partial_x f_j(x, y')| > \tau$.

Proof. Fix $y \in \mathcal{Y}^0$. There exist, possibly degenerate, intervals $I_{j,k} = I_{j,k}(y) \subset [0, 1]$ such that $|\partial_x f_j(x, y)| \leq 2\tau$ for $x \in \cup_k I_{j,k}$ and $|\partial_x f_j(x, y)| > 2\tau$ for $x \in [0, 1] \setminus (\cup_k I_{j,k})$. We let $\mathcal{Z}_{j,k} = \{f_j(x, y) : x \in I_{j,k}\}$, $\mathcal{Z}_y = \cup_{j,k} \mathcal{Z}_{j,k}$, and we define

$$\mathcal{Z}_y := \left\{ z \in \mathcal{Z}^0 : \text{dist}\left(z, \mathcal{Z}_y \cup \mathcal{Z}_y^0 \cup (\mathcal{Z}^0)^c\right) \leq \tau \right\}.$$

Suppose that $f_j(x, y') \in \mathcal{Z}^0 \setminus \mathcal{Z}_y$, for some $y' \in (y - \delta, y + \delta)$. By (6.2) and $\delta \leq \tau/C_1$, it follows that $f_j(x, y) \in \mathcal{Z}^0 \setminus (\mathcal{Z}_y \cup \mathcal{Z}_y^0)$. Hence $|\partial_x f_j(x, y)| > 2\tau$, and by (6.3) and $\delta \leq r_0, \tau/C_3$, it follows that $|\partial_x f_j(x, y')| > \tau$, as desired.

We clearly have that $\text{mes}(\mathcal{Z}_{j,k}) \leq \tau \text{mes}(I_{j,k})$, and hence $\text{mes}(\mathcal{Z}_y) \leq n\tau$. At the same time we have

$$\text{mes}(\mathcal{Z}_y) \leq \text{mes}(\mathcal{Z}_y) + \text{mes}(\mathcal{Z}_y^0) + 2\tau(\text{compl}(\mathcal{Z}_y) + \text{compl}(\mathcal{Z}_y^0) + 2),$$

$$\text{compl}(\mathcal{Z}_y) \leq \text{compl}(\mathcal{Z}_y) + \text{compl}(\mathcal{Z}_y^0) + 2$$

(recall that \mathcal{Z}^0 is an interval). So to get (6.4) we just need to estimate the number of intervals $I_{j,k}$. The number of these intervals is controlled by the number of solutions of $\partial_x f_j(x, y) = \pm 2\tau$, $j = 1, \dots, n$ which is bounded by the number of solutions of the system

$$\begin{aligned} 0 &= Q_1(e(x), z) := P(e(x), e(y), z) \\ 0 &= Q_2(e(x), z) := \partial_1 P(e(x), e(y), z) 2\pi i e(x) \pm 2\tau \partial_3 P(e(x), e(y), z). \end{aligned}$$

By Bézout's Theorem it follows that the number of solutions of the above system is controlled by d_2^2 . This concludes the proof. \square

Let us make some remarks regarding the use of Bézout's Theorem in the above lemma. To apply the theorem we would want Q_1 and Q_2 to be irreducible and distinct. They are not necessarily irreducible but we can replace them with some irreducible factors by the following simple observation. Since $Q_i(e(x), f_j(x, y)) = 0$ and f_j is analytic, there must exist an irreducible factor \tilde{Q}_i of Q_i such that $\tilde{Q}_i(e(x), f_j(x, y)) = 0$. We can ensure that \tilde{Q}_1 and \tilde{Q}_2 are different by varying τ . Of course, for different functions f_j we may get different irreducible factors. It is elementary to argue that when we add up the numbers of solutions from each combination of irreducible factors we get a number less than the product of the degrees of Q_1 and Q_2 . In what follows, similar considerations apply whenever we use Bézout's Theorem.

We can now obtain elimination of resonances.

Theorem 6.2. *Let $\tau, \sigma > 0$, $Q \geq \max\{4C_1/\tau, d_1, n(d_2^2 + C_2)\}$, $M \geq Q$, $\delta \leq \min\{r_0, \tau/C_3, \tau/C_1\}$, $\delta' \leq \min\{\sigma/(MC_0 + 2C_1), \delta/2\}$. There exists $\mathcal{Y} \subset [0, 1]$, with*

$$\text{mes}(\mathcal{Y}) \lesssim \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}, \text{compl}(\mathcal{Y}) \lesssim \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}/\delta', \quad (6.5)$$

such that for each $y \in \mathcal{Y}^0 \setminus \mathcal{Y}$ there exists $\tilde{\mathcal{Z}}_y$, with

$$\begin{aligned} \text{mes}(\tilde{\mathcal{Z}}_y) &\lesssim n\tau + c_0 + (d_2^2 + C_2)(\tau + \sigma) + C_0 \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}, \\ \text{compl}(\tilde{\mathcal{Z}}_y) &\lesssim M d_2 (d_2^2 + C_2), \end{aligned}$$

such that for any $x \in \mathbb{R}$ we have that if $f_j(x, y) \in \mathcal{Z}^0 \setminus \tilde{\mathcal{Z}}_y$, for some j , then

$$|f_j(x, y) - f_k(x + my, y)| \geq \sigma, \quad (6.6)$$

for $k = 1, \dots, n$ and any integer m , $Q \leq |m| \leq M$.

Proof. Let $\{y_\alpha\}$ be a δ -net of points from \mathcal{Y}^0 covering \mathcal{Y}^0 . Also let $I_{y_\alpha} = (y_\alpha - \delta, y_\alpha + \delta)$, and $\mathcal{Z}'_{y_\alpha} = \left\{ z \in \mathcal{Z}^0 : \text{dist} \left(\mathcal{Z}_{y_\alpha} \cup (\mathcal{Z}^0)^C \right) \leq \sigma \right\}$, where \mathcal{Z}_{y_α} is as in Lemma 6.1. By (6.4), there exists a union of intervals $Z_{y_\alpha} \supset \mathcal{Z}'_{y_\alpha}$ such that

$$\text{mes}(Z_{y_\alpha}) \lesssim n\tau + c_0 + (d_2^2 + C_2)(\tau + \sigma), \text{compl}(Z_{y_\alpha}) \lesssim d_2^2 + C_2.$$

Let

$$\mathcal{B}(y_\alpha, j) = \left\{ (x, y) \in [0, 1] \times I_{y_\alpha} : f_j(x, y) \in (\mathcal{Z}^0)^C \cup Z_{y_\alpha} \right\}.$$

We define

$$g_{j,k,m}(x, y) = f_k(x + my, y) - f_j(x, y),$$

$$\mathcal{B}'_m(y_\alpha, j, k) = \{(x, y) \in ([0, 1] \times I_{y_\alpha}) \setminus \mathcal{B}(y_\alpha, j) : |g_{j,k,m}(x, y)| < \sigma\},$$

and $\mathcal{B}'_m(y_\alpha) = \cup_{j,k} \mathcal{B}'_m(y_\alpha, j, k)$. For $(x, y) \in \mathcal{B}'_m(y_\alpha, j, k)$ we have that $f_j(x, y) \in \mathcal{Z}^0 \setminus Z_{y_\alpha}$ (due to the definition of $\mathcal{B}(y_\alpha, j)$) and consequently $f_k(x + my, y) \in \mathcal{Z}^0 \setminus \mathcal{Z}_{y_\alpha}$ (due to the definitions of Z_{y_α} and \mathcal{Z}'_{y_α}). Hence, by Lemma 6.1, for $(x, y) \in \mathcal{B}'_m(y_\alpha, j, k)$ we have $|\partial_x f_k(x + my, y)| > \tau$. Since

$$\partial_y g_{j,k,m}(x, y) = m \partial_x f_k(x + my, y) + \partial_y f_k(x + my, y) - \partial_y f_j(x, y)$$

we can conclude that $|\partial_y g_{j,k,m}(x, y)| \geq |m|\tau - 2C_1 \geq |m|\tau/2$ for $(x, y) \in \mathcal{B}'_m(y_\alpha, j, k)$ (we used (6.1) and $|m| \geq Q \geq 4C_1/\tau$). Let

$$\mathcal{B}''_m(y_\alpha, j, k) = \{(x, y) \in [0, 1] \times I_{y_\alpha} : |g_{j,k,m}(x, y)| < \sigma\}.$$

For a set $S \subset \mathbb{R}^2$ we will use the notation $S|_x := \{y : (x, y) \in S\}$, $S|_y := \{x : (x, y) \in S\}$. We have that $\mathcal{B}'_m(y_\alpha, j, k)|_x = \mathcal{B}''_m(y_\alpha, j, k)|_x \setminus \mathcal{B}(y_\alpha, j)|_x$ is a union of, possibly degenerate, intervals. On each such interval we have $\partial_y g_{j,k,m}(x, \cdot) \geq |m|\tau/2$ or $\partial_y g_{j,k,m}(x, \cdot) \leq -|m|\tau/2$, so by the fundamental theorem of calculus and the fact that on these intervals we have $|g_{j,k,m}(x, \cdot)| < \sigma$, each such interval must be of size smaller than $2\sigma(|m|\tau)^{-1}$. Consequently we get

$$\text{mes}(\mathcal{B}'_m(y_\alpha)|_x) \leq 2\sigma(|m|\tau)^{-1} \sum_{j,k} \text{compl}(\mathcal{B}'_m(y_\alpha, j, k)|_x). \quad (6.7)$$

At the same time we have

$$\text{compl}(\mathcal{B}'_m(y_\alpha, j, k)|_x) \leq \text{compl}(\mathcal{B}''_m(y_\alpha, j, k)|_x) + \text{compl}(\mathcal{B}(y_\alpha, j)|_x).$$

The total number of components in $\mathcal{B}(y_\alpha, j)|_x$ for all j is controlled by the number of solutions of

$$\begin{cases} f_j(x, y) = z \\ j \in \{1, \dots, n\} \\ z \in E((\mathcal{Z}^0)^C \cup Z_{y_\alpha}) \end{cases},$$

where $E\left((\mathcal{Z}^0)^C \cup Z_{y_\alpha}\right)$ is the set consisting of the endpoints of the intervals in $(\mathcal{Z}^0)^C \cup Z_{y_\alpha}$. The number of solutions of this system is bounded by the number of solutions of

$$\begin{cases} 0 = Q_1(e(y), z) := P(e(x), e(y), z) \\ 0 = Q_2(e(y), z) := z - z' \\ z' \in E\left((\mathcal{Z}^0)^C \cup Z_{y_\alpha}\right) \end{cases}.$$

Using Bézout's theorem, we can conclude that

$$\sum_j \text{compl}(\mathcal{B}(y_\alpha, j)) \lesssim d_1(d_2^2 + C_2).$$

The total number of components in $\mathcal{B}''(y_\alpha, j, k)|_x$ for all j, k , is controlled by the number of solutions of

$$\begin{cases} g_{j,k,m}(x, y) = \pm\sigma \\ j, k \in \{1, \dots, n\} \end{cases},$$

which is bounded by the number of solutions of

$$\begin{cases} 0 = Q_1(e(y), z) := P(e(x), e(y), z) \\ 0 = Q_2(e(y), z) := e(|m|d_2y)P(e(x+my), e(y), z \pm \sigma) \end{cases}.$$

The $e(|m|d_2y)$ factor ensures that Q_2 is a polynomial, even when $m < 0$. Since $\deg Q_1 \leq d_1$ and $\deg Q_2 \lesssim |m|d_2 + d_1$, using Bézout's theorem we can conclude that

$$\sum_{j,k} \text{compl}(\mathcal{B}_m''(y_\alpha, j, k)|_x) \lesssim |m|d_2d_1 + d_1^2 \lesssim |m|d_2d_1$$

(we used $|m| \geq Q \geq d_1$). Now we can conclude that

$$\sum_{j,k} \text{compl}(\mathcal{B}_m'(y_\alpha, j, k)|_x) \lesssim |m|d_2d_1 + nd_1(d_2^2 + C_2) \lesssim |m|d_2d_1$$

(we used $|m| \geq Q \geq n(d_2^2 + C_2)$). By (6.7) and Fubini's theorem we can now conclude that $\text{mes}(\mathcal{B}_m'(y_\alpha)) \lesssim \sigma d_2 d_1 \tau^{-1}$.

Let $\mathcal{B}' = \cup_{m, y_\alpha} \mathcal{B}_m'(y_\alpha)$. We have that $\text{mes}(\mathcal{B}') \lesssim M \sigma d_2 d_1 \tau^{-1} \delta^{-1}$. We define

$$\mathcal{Y} := \left\{ y \in \mathcal{Y}^0 : \text{mes}(\mathcal{B}'|_y) > \sqrt{M \sigma d_2 d_1 \tau^{-1} \delta^{-1}} \right\}.$$

From Chebyshev's inequality we get $\text{mes}(\mathcal{Y}) \lesssim \sqrt{M \sigma d_2 d_1 \tau^{-1} \delta^{-1}}$. Clearly, for $y \in \mathcal{Y}_0 \setminus \mathcal{Y}$ we have

$$\text{mes}(\mathcal{B}'|_y) \leq \sqrt{M \sigma d_2 d_1 \tau^{-1} \delta^{-1}}. \quad (6.8)$$

Next we estimate the complexity of the set \mathcal{Y} . Cover \mathcal{Y} by $\lesssim \sqrt{M \sigma d_2 d_1 \tau^{-1} \delta^{-1}}/\delta'$ intervals Y_i of size δ' , centered at points from \mathcal{Y} . Suppose that y_i is the center of Y_i . Since $y_i \in \mathcal{Y}$ we have that there exists m , $Q \leq |m| \leq M$, such that

$$|f_k(x + m y_i, y_i) - f_j(x, y_i)| < \sigma, x \in \mathcal{B}'|_{y_i}.$$

Due to (6.2) we can conclude that

$$|f_k(x+my, y) - f_j(x, y)| \leq |f_k(x+my_i, y_i) - f_j(x, y_i)| + (MC_0 + 2C_1)|y - y_i| < 2\sigma,$$

for $x \in \mathcal{B}'|_{y_i}$, and $y \in Y_i$ (we used $\delta' \leq \sigma/(MC_0 + 2C_1)$). From this we get that

$$\cup_i Y_i \subset \left\{ y \in \tilde{\mathcal{Y}}^0 : \text{mes}(\mathcal{B}'(2\sigma)|_y) > \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}} \right\},$$

where $\mathcal{B}'(2\sigma)$ has the same definition as \mathcal{B}' , only with 2σ instead of σ , and

$$\tilde{\mathcal{Y}}^0 = \{y \in [0, 1] : \text{dist}(y, \mathcal{Y}^0) \leq \delta/2\}$$

(we used $\delta' \leq \delta/2$). Note that the δ -net $\{y_\alpha\}$ can be chosen so that it covers $\tilde{\mathcal{Y}}^0$, rather than just \mathcal{Y}^0 . By the same argument as above (that led to $\text{mes}(\mathcal{Y}) \lesssim \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}$) we get that $\text{mes}(\cup_i Y_i) \lesssim \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}$, and hence we have (6.5).

Fix $y \in \mathcal{Y}^0 \setminus \mathcal{Y}$ and let y_α be such that $y \in I_{y_\alpha}$. Let

$$Z'_y = \cup_{m,j,k} \{f_j(x, y) : (x, y) \in \mathcal{B}'_m(y_\alpha, j, k)\},$$

and define $\tilde{Z}_y := Z_{y_\alpha} \cup Z'_y$. We have that

$$\begin{aligned} \text{mes}(\tilde{Z}_y) &\leq \text{mes}(Z_{y_\alpha}) + \text{mes}(Z'_y) \\ &\lesssim n\tau + c_0 + (d_2^2 + C_2)(\tau + \sigma) + C_0 \sqrt{M\sigma d_2 d_1 \tau^{-1} \delta^{-1}}, \end{aligned}$$

$$\begin{aligned} \text{compl}(\tilde{Z}_y) &\leq \text{compl}(Z_{y_\alpha}) + \text{compl}(Z'_y) \lesssim d_2^2 + C_2 + M(d_2^2 + d_2(d_2^2 + C_2)) \\ &\lesssim Md_2(d_2^2 + C_2). \end{aligned}$$

To get the bound on $\text{mes}(Z'_y)$ we used (6.8) and (6.2). The estimate on $\text{compl}(Z'_y)$ is obtained by noticing that

$$\text{compl}(Z'_y) \leq \sum_{j,k,m} \text{compl}(\mathcal{B}'_m(y_\alpha, j, k)|_y),$$

and by using Bézout's theorem in the same way we did to estimate

$$\sum_{j,k} \text{compl}(\mathcal{B}'_m(y_\alpha, j, k)|_x).$$

It is easy to see that with this choice of \tilde{Z}_y we have that (6.6) holds. Indeed, suppose that $f_j(x, y) \in \mathcal{Z}^0 \setminus \tilde{Z}_y$ and suppose that there exist k, m , such that

$$|f_j(x, \omega) - f_k(x + m\omega, \omega)| < \sigma.$$

This implies $(x, y) \in \mathcal{B}''_m(y_\alpha, j, k) \subset \mathcal{B}'_m(y_\alpha, j, k) \cup \mathcal{B}(y_\alpha, j)$. If $(x, y) \in \mathcal{B}'_m(y_\alpha, j, k)$ then $f_j(x, y) \in Z'_y$, and if $(x, y) \in \mathcal{B}(y_\alpha, j)$ then $f_j(x, y) \in Z_{y_\alpha} \cup (\mathcal{Z}^0)^C$. Either way, we arrived at a contradiction. This concludes the proof. \square

7 Elimination of Resonances and Separation of Eigenvalues via Slopes

In this section we apply Theorem 6.2 to our concrete setting to obtain a sharper elimination of resonances, based on which we will obtain our main result (by applying Theorem 4.4). As was mentioned in the introduction, to get the stability of slopes needed for Theorem 6.2 we will at first use the “a priori” separation via resultants (Proposition 5.7). This will yield a better separation, but still weaker than the one we desire (see Proposition 7.4). By using the improved separation to get better stability of slopes and then repeating our steps we will obtain the desired separation.

We proceed by setting things up for the use of Theorem 6.2. First, we need to approximate a and b by trigonometric polynomials, so that the eigenvalues will be algebraic. Let

$$a(x) = \sum_{n=-\infty}^{\infty} a_n e(nx),$$

and

$$b(x) = \sum_{n=-\infty}^{\infty} b_n e(nx),$$

be the Fourier series expansions for a and b (recall that $e(x) = \exp(2\pi i x)$). It is known that there exist constant $C = C(\|a\|_{\infty}, \|b\|_{\infty})$ and $c = c(\rho_0)$ such that

$$|a_n|, |b_n| \leq C \exp(-\pi \rho_0 |n|), n \in \mathbb{Z}, \quad (7.1)$$

with $C = \sup_{x \in \mathbb{T}} (|a(x \pm i\rho_0/2)| + |b(x \pm i\rho_0/2)|)$. Let

$$a_K(x) = \sum_{n=-K}^K a_n e(nx),$$

and

$$b_K(x) = \sum_{n=-K}^K b_n e(nx).$$

By (7.1), there exists $C = C(\|a\|_{\infty}, \|b\|_{\infty}, \rho_0)$ such that

$$\sup_{|\operatorname{Im} z| < \rho_0/3} |a(z) - a_K(z)| + |b(z) - b_K(z)| \leq C \exp(-\pi \rho_0 K/3). \quad (7.2)$$

Let $H_K^{(l)}(x, \omega)$ denote the matrix, at scale l , associated with a_K, b_K , and let $E_{K,j}^{(l)}(x, \omega)$ be its eigenvalues. As a consequence of (7.2) we get

$$\sup_{x, \omega \in \mathbb{T}} \left\| H^{(l)}(x, \omega) - H_K^{(l)}(x, \omega) \right\| \lesssim C \exp(-cK),$$

and, since the matrices are Hermitian for $x, \omega \in \mathbb{T}$, we also have

$$\sup_{x, \omega \in \mathbb{T}} \left| E_j^{(l)}(x, \omega) - E_{K,j}^{(l)}(x, \omega) \right| \lesssim C \exp(-cK). \quad (7.3)$$

It is easy to see that there exists a constant $C = C(\|a\|_\infty, \|b\|_*, \rho_0)$ such that

$$\|H^{(l)}(z, w) - H^{(l)}(z', w')\| \leq C(|z - z'| + l|w - w'|),$$

for any $z, z' \in \mathbb{H}_{\rho_0/3}$ and $w, w' \in l^{-1}\mathbb{H}_{\rho_0/3}$. Furthermore, due to (7.2), it can be seen that there exists a constant $C = C(\|a\|_\infty, \|b\|_*, \rho_0)$ such that for any K we have

$$\|H_K^{(l)}(z, w) - H_K^{(l)}(z', w')\| \leq C(|z - z'| + l|w - w'|), \quad (7.4)$$

for any $z, z' \in \mathbb{H}_{\rho_0/3}$ and $w, w' \in l^{-1}\mathbb{H}_{\rho_0/3}$. In particular, since $H_K^{(l)}(x, \omega)$ is Hermitian for $x, \omega \in \mathbb{T}$, we have that

$$|E_{K,j}^{(l)}(x, \omega) - E_{K,j}^{(l)}(x', \omega')| \leq C(|x - x'| + l|\omega - \omega'|), \quad (7.5)$$

for any $x, \omega \in \mathbb{T}$. This will give us the values of the constants in (6.2). To get the constants related to (6.3) we will use the following lemma.

Lemma 7.1. *Fix $x, \omega \in \mathbb{T}$, $j \in \{1, \dots, l\}$, and suppose that $|E_j^{(l)}(x, \omega) - E_i^{(l)}(x, \omega)| \geq \sigma$, for all $i \neq j$. Furthermore, suppose that K is large enough so that*

$$|E_i^{(l)}(x, \omega) - E_{K,i}^{(l)}(x, \omega)| \leq \sigma/2$$

for all i . There exists a constant $C_0 = C_0(\|a\|_\infty, \|b\|_, \rho_0)$ such that*

$$|\partial_x E_{K,j}^{(l)}(x, \omega) - \partial_x E_{K,j}^{(l)}(x, \omega')| \leq C_0 l \sigma^{-1} |\omega - \omega'|,$$

for any $\omega' \in \mathbb{R}$ such that $|\omega - \omega'| \leq C_0^{-1} l^{-1} \sigma$.

Proof. We clearly have that

$$|E_{K,j}^{(l)}(x, \omega) - E_{K,i}^{(l)}(x, \omega)| \geq \sigma/2.$$

From (7.4) and standard perturbation theory it follows that

$$|E_{K,j}^{(l)}(z, w) - E_{K,i}^{(l)}(z, w)| \geq \sigma/4, \quad (7.6)$$

for any $i \neq j$ and $(z, w) \in \mathcal{D}(x, c\sigma) \times \mathcal{D}(\omega, c\sigma/l) =: \mathcal{P}$. We can choose $c = c(\|a\|_\infty, \|b\|_*, \rho_0)$ small enough so that we also have

$$\|H_K^{(l)}(z, w) - H_K^{(l)}(x, \omega)\| \leq C(|z - x| + l|w - \omega|) \leq \sigma/8, \quad (7.7)$$

for any $(z, w) \in \mathcal{P}$. Since $E_{K,j}^{(l)}$ is simple on \mathcal{P} it follows from the implicit function theorem that it is analytic on \mathcal{P} . From (7.7) it follows that given $(z, w) \in \mathcal{P}$ we have

$$|E_{K,j}^{(l)}(z, w) - E_{K,j'}^{(l)}(x, \omega)| \leq C(|z - x| + l|w - \omega|) \leq \sigma/8,$$

for some $j' = j'(z, w)$. Due to (7.6) and the continuity of $E_{K,j}^{(l)}$ it follows that in fact for $(z, w) \in \mathcal{P}$ we have

$$|E_{K,j}^{(l)}(z, w) - E_{K,j}^{(l)}(x, \omega)| \leq C(|z - x| + l|w - \omega|).$$

This estimate and Cauchy's formula yield the desired conclusion. \square

Corollary 7.2. Fix $A > 1$ and let $l = 2 \left\lceil (\log N)^A \right\rceil$. There exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, A)$ such that for $N \geq N_0$ there exists Ω_l , with

$$\text{mes}(\Omega_l) < \exp\left(-(\log \log N)^2\right), \text{compl}(\Omega_l) < (\log N)^{2A+1},$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_l$ there exists a set $\mathcal{E}_{l,\omega}$, with

$$\text{mes}(\mathcal{E}_{l,\omega}) < \exp\left(-(\log \log N)^2\right), \text{compl}(\mathcal{E}_{l,\omega}) < (\log N)^{2A+1},$$

such that for any $x \in \mathbb{T}$, $K \geq (\log N)^{1/2}$, if $E_{K,j}^{(l)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{l,\omega}$, for some j , then

$$\left| \partial_x E_{K,j}^{(l)}(x, \omega) - \partial_x E_{K,j}^{(l)}(x, \omega') \right| \leq \exp\left((\log N)^{1/2}\right) |\omega - \omega'|,$$

for any $\omega' \in \mathbb{R}$ such that $|\omega - \omega'| \leq \exp\left(-(\log N)^{1/2}\right)$.

Proof. The result follows immediately from Lemma 7.1 and Proposition 5.7 with $\delta = 1/3A$. \square

Let $f_{K,l}(x, \omega, E) = \det \left[H_K^{(l)}(x, \omega) - E \right]$. It is straightforward to see that

$$P(e(x), e(\omega), E) := e(20Klx)e(20Kl^2\omega)f_{K,l}(x, \omega, E)f_{K,l+1}(x, \omega, E) \cdot f_{K,2l}(x, \omega, E)f_{K,2l+1}(x, \omega, E) \quad (7.8)$$

is a polynomial of degree $\lesssim Kl^2$ when the first variable is fixed, and of degree $\lesssim Kl$ when the second variable is fixed. Let $K_0 = C \lceil \log N \rceil$, where $C = C(\|a\|_\infty, \|b\|_*, \rho_0)$ is chosen such that

$$\sup_{x, \omega \in \mathbb{T}} \left| E_j^{(l)}(x, \omega) - E_{K_0,j}^{(l)}(x, \omega) \right| \leq \frac{1}{N^2} \quad (7.9)$$

(we used (7.3)).

We can now apply Theorem 6.2.

Proposition 7.3. Fix $A > 1$, $p \in (1, 2)$, and let $l = 2 \left\lceil (\log N)^A \right\rceil$. There exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, A, p)$, such that for any $N \geq N_0$ there exists a set Ω_N , with

$$\text{mes}(\Omega_N) < \exp\left(-(\log \log N)^2/2\right), \text{compl}(\Omega_N) < N^{1+p},$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ there exists a set $\mathcal{E}_{N,\omega}$, with

$$\text{mes}(\mathcal{E}_{N,\omega}) < (\log N)^{-A}, \text{compl}(\mathcal{E}_{N,\omega}) < N(\log N)^{4A},$$

such that for any $x \in \mathbb{T}$ and any integer m , $(\log N)^{6A} \leq |m| \leq N$, we have

$$\text{dist}\left(\mathcal{E}^0 \cap \text{spec}\left(H^{(l_1)}(x, \omega) \setminus \mathcal{E}_{N,\omega}\right), \text{spec}\left(H^{(l_2)}(x + m\omega, \omega)\right)\right) \geq \frac{2}{N^p},$$

$$l_1, l_2 \in \{l, l+1, 2l, 2l+1\}.$$

Proof. We begin by identifying all the parameters used in Section 6. The polynomial P is given by (7.8), and we can take $d_1 = C(\log N)^{2A+1}$, $d_2 = C(\log N)^{A+1}$, with $C = C(\|a\|_\infty, \|b\|_*, \rho_0)$. We have $\{f_j\} = \left\{E_{K_0,i}^{(l')}: i \in \{1, \dots, l'\}, l' \in \{l, l+1, 2l, 2l+1\}\right\}$, and $n = 6l+2$. By (7.5) we can choose $C_0 = C$, $C_1 = C(\log N)^A$, $C = C(\|a\|_\infty, \|b\|_*, \rho_0)$. Let $\Omega_l, \mathcal{E}_{l,\omega}$ be as in Corollary 7.2. By Corollary 7.2 we can choose $\mathcal{Y}^0 = \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_l$, $\mathcal{Z}^0 = \mathcal{E}^0$, $\mathcal{Z}_y^0 = \mathcal{E}_{l,\omega}$, $c_0 = \exp(-(\log \log N)^2)$, $C_2 = (\log N)^{2A+1}$, $C_3 = \exp((\log N)^{1/2})$, $r_0 = \exp(-(\log N)^{1/2})$.

Next we apply Theorem 6.2 with $\tau = (\log N)^{-5A}$, $\sigma = 4N^{-p}$, $Q = (\log N)^{6A}$, $M = N$, $\delta = \exp(-(\log N)^{2/3})$, $\delta' = cN^{-(1+p)}$, $c = c(\|a\|_\infty, \|b\|_*, \rho_0)$. Let $\Omega_{K_0,l} = \Omega_l \cup \mathcal{Y}$ and $\mathcal{E}_{K_0,l,\omega} = \tilde{\mathcal{Z}}_y$. We have

$$\text{mes}(\Omega_{K_0,l}) < \exp(-(\log \log N)^2/2), \text{compl}(\Omega_{K_0,l}) < N^{1+p},$$

$$\text{mes}(\mathcal{E}_{K_0,l,\omega}) < (\log N)^{-2A}, \text{compl}(\mathcal{E}_{K_0,l,\omega}) < N(\log N)^{4A},$$

and

$$\text{dist}\left(\mathcal{E}^0 \cap \text{spec}\left(H_{K_0}^{(l_1)}(x, \omega) \setminus \mathcal{E}_{K_0,l,\omega}\right), \text{spec}\left(H_{K_0}^{(l_2)}(x + m\omega, \omega)\right)\right) \geq \frac{4}{N^p}, \quad (7.10)$$

$l_1, l_2 \in \{l, l+1, 2l, 2l+1\}$, for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_{K_0,l}$, any $x \in \mathbb{T}$, and any integer m , $(\log N)^{6A} \leq |m| \leq N$.

Let $\Omega_N = \Omega_{K_0,l}$, and $\mathcal{E}_{N,\omega} = \left\{E \in \mathcal{E}^0 : \text{dist}\left(E, \mathcal{E}_{K_0,l,\omega} \cup (\mathcal{E}^0)^c\right) \leq N^{-2}\right\}$. The measure and complexity bounds for Ω_N and $\mathcal{E}_{N,\omega}$ are clearly satisfied. Fix $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$ and $x \in \mathbb{T}$, and suppose $E_j^{(l_1)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N,\omega}$, for some j and $l_1 \in \{l, l+1, 2l, 2l+1\}$. By (7.9) it follows that $E_{K_0,j}^{(l_1)} \in \mathcal{E}^0 \setminus \mathcal{E}_{K_0,l,\omega}$. Hence the conclusion follows from (7.10) and (7.9). \square

We can now improve the separation of eigenvalues at scale N by applying Theorem 4.4.

Proposition 7.4. Fix $p \in (1, 2)$. There exist constants $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, p)$, $C_0 = C_0(\alpha)$ such that for any $N \geq N_0$ there exists a set $\tilde{\Omega}_N$, with

$$\text{mes}(\tilde{\Omega}_N) \leq \exp(-(\log \log N)^2/4), \text{compl}(\tilde{\Omega}_N) \lesssim N^{1+p},$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \tilde{\Omega}_N$ there exists a set $\tilde{\mathcal{E}}_{N,\omega}$, with

$$\text{mes}(\tilde{\mathcal{E}}_{N,\omega}) \leq (\log N)^{-1/10}, \text{compl}(\tilde{\mathcal{E}}_{N,\omega}) \leq N(\log N)^{C_0},$$

such that for any $x \in \mathbb{T}$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \tilde{\mathcal{E}}_{N,\omega}$, for some j , then

$$\left|E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega)\right| > \frac{1}{N^p},$$

for any $k \neq j$.

Proof. We start by identifying the parameters from the Elimination Assumption 3.1. Apply Proposition 7.3 with $A = A(\alpha)$ as in the Elimination Assumption 3.1. Now we can choose $\Omega_N, \mathcal{E}_{N,\omega}$ as in Proposition 7.3 and we also have $Q_N = (\log N)^{6A}, \sigma_N = 2N^{-p}$.

Next we apply Theorem 4.4 with $N' = \left\lceil \exp\left((\log N)^{1/7A}\right) \right\rceil$. The conclusion follows by setting $\tilde{\Omega}_N = \Omega_N \cup \Omega_{N'}$, and

$$\tilde{\mathcal{E}}_{N,\omega} = \left\{ E \in \mathcal{E}^0 : \text{dist}\left(\mathcal{E}_{N,\omega} \cup \mathcal{E}_{N',\omega} \cup (\mathcal{E}^0)^C\right) < 2N^{-p} \right\}.$$

□

We can now repeat our steps, starting with Corollary 7.2 to obtain a better separation. This time we will eliminate the resonances at scale $l = 100[(\log N)/\gamma]$ and then use localization to eliminate the resonances at the scale $l = 2\left\lceil (\log N)^A \right\rceil$, as needed to apply Theorem 4.4. Working at a scale $l = C[\log N]$ is needed to get separation by $(N(\log N)^p)^{-1}$ with p as small as possible. We need to have $C = C(\gamma)$ in order to be able to apply localization.

Lemma 7.5. *Fix $p \in (1, 2)$ and let $l = 100[(\log N)/\gamma]$. There exist constants $C_0 = C_0(\alpha)$, $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, \mathcal{E}^0, p)$, such that for $N \geq N_0$ there exists Ω_l , with*

$$\text{mes}(\Omega_l) \leq \exp\left(-(\log \log \log N)^2/8\right), \text{compl}(\Omega_l) \lesssim (\log N/\gamma)^{1+p},$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_l$ there exists a set $\mathcal{E}_{l,\omega}$, with

$$\text{mes}(\mathcal{E}_{l,\omega}) \lesssim (\log \log N)^{-1/10}, \text{compl}(\mathcal{E}_{l,\omega}) \leq \log N (\log \log N)^{C_0},$$

such that for any $x \in \mathbb{T}$, if $E_{K_0,j}^{(l)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{l,\omega}$, for some j , then

$$\left| \partial_x E_{K_0,j}^{(l)}(x, \omega) - \partial_x E_{K_0,j}^{(l)}(x, \omega') \right| \lesssim (\log N)^{1+p} \gamma^{-1} |\omega - \omega'|,$$

for any $\omega' \in \mathbb{R}$ such that $|\omega - \omega'| \leq (\log N)^{-(1+p)} \gamma$.

Proof. The result follows immediately from Lemma 7.1 and Proposition 7.4. □

We can now apply Theorem 6.2 again.

Proposition 7.6. *Fix $\tilde{p} > 15$, and let $l = 100[(\log N)/\gamma]$. There exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, \tilde{p})$ such that for any $N \geq N_0$ there exists a set Ω_N , with*

$$\text{mes}(\Omega_N) < \exp\left(-(\log \log \log N)^2/10\right), \text{compl}(\Omega_N) < N^2 (\log N)^{\tilde{p}},$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_l$ there exists a set $\mathcal{E}_{N,\omega}$, with

$$\text{mes}(\mathcal{E}_{N,\omega}) \lesssim (\log \log N)^{-1/10}, \text{compl}(\mathcal{E}_{N,\omega}) \lesssim N (\log N)^6,$$

such that for any $x \in \mathbb{T}$ and any integer m , $(\log N)^6 \leq |m| \leq 2N$, we have

$$\text{dist}\left(\mathcal{E}^0 \cap \text{spec}\left(H^{(l)}(x, \omega) \setminus \mathcal{E}_{N,\omega}\right), \text{spec}\left(H^{(l)}(x + m\omega, \omega)\right)\right) \geq \frac{3}{N(\log N)^{\tilde{p}}}.$$

Proof. We begin by identifying all the parameters used in Section 6. The polynomial P is given by

$$P(e(x), e(\omega), E) = e(K_0 l x) e(K_0 l^2 \omega) f_{K_0, l}(x, \omega, E),$$

and we can take $d_1 = C(\log N)^3$, $d_2 = C(\log N)^2$, with $C = C(\|a\|_\infty, \|b\|_*, \rho_0, \gamma)$. We have $\{f_j\} = \{E_{K_0, i}^{(l)} : i \in \{1, \dots, l\}\}$, and $n = l$. By (7.5) we can choose $C_0 = C$, $C_1 = C \log N$, with $C = C(\|a\|_\infty, \|b\|_*, \rho_0, \gamma)$. Let Ω_l , $\mathcal{E}_{l, \omega}$ be as in Lemma 7.5. By Lemma 7.5, with $p \in (1, 2)$ such that $14 + p < \tilde{p}$, we can choose $\mathcal{Y}^0 = \mathbb{T}_{c, \alpha} \setminus \Omega_l$, $\mathcal{Z}^0 = \mathcal{E}^0$, $\mathcal{Z}_y^0 = \mathcal{E}_{l, \omega}$, $c_0 = (\log \log N)^{-1/10}$, $C_2 = \log N (\log \log N)^C$, with $C = C(\alpha)$, $C_3 = (\log N)^{1+p} \gamma^{-1}$, $r_0 = 1/C_3$.

Next we apply Theorem 6.2 with $\tau = (\log N)^{-4} (\log \log N)^{-1}$, $\sigma = 4N^{-1} (\log N)^{-\tilde{p}}$, $Q = (\log N)^6$, $M = 2N$, $\delta = (\log N)^{-(5+p)} (\log \log N)^{-2}$, $\delta' = cN^{-2} (\log N)^{-\tilde{p}}$, with $c = c(\|a\|_\infty, \|b\|_*, \rho_0, \gamma)$, $M = 2N$. Let $\Omega_{K_0, l} = \Omega_l \cup \mathcal{Y}$ and $\mathcal{E}_{K_0, l, \omega} = \tilde{\mathcal{Z}}_y$. We have

$$\text{mes}(\Omega_{K_0, l}) \leq \exp(-(\log \log \log N)^2 / 10), \text{compl}(\Omega_{K_0, l}) \leq N^2 (\log N)^{\tilde{p}},$$

$$\text{mes}(\mathcal{E}_{K_0, l, \omega}) \lesssim (\log \log N)^{-1/10}, \text{compl}(\mathcal{E}_{K_0, l, \omega}) \lesssim N (\log N)^6,$$

and

$$\text{dist}\left(\mathcal{E}^0 \cap \text{spec}\left(H_{K_0}^{(l)}(x, \omega) \setminus \mathcal{E}_{K_0, l, \omega}\right), \text{spec}\left(H_{K_0}^{(l)}(x + m\omega, \omega)\right)\right) \geq \frac{4}{N (\log N)^{\tilde{p}}},$$

for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_{K_0, l}$, any $x \in \mathbb{T}$ and any integer m , $(\log N)^6 \leq |m| \leq 2N$.

The conclusion follows just as in the proof of Proposition 7.3, by setting $\Omega_N = \Omega_{K_0, l}$, and $\mathcal{E}_{N, \omega} = \left\{E \in \mathcal{E}^0 : \text{dist}\left(E, \mathcal{E}_{K_0, l, \omega} \cup (\mathcal{E}^0)^C\right) \leq N^{-2}\right\}$. \square

Next we obtain the new version of Proposition 7.3.

Proposition 7.7. Fix $p > 15$, $A > 1$ and let $l = 2 \left\lceil (\log N)^A \right\rceil$. There exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, p, A)$, such that for any $N \geq N_0$ there exists a set Ω_N , with

$$\text{mes}(\Omega_N) \lesssim \exp(-(\log \log \log N)^2 / 10), \text{compl}(\Omega_N) \lesssim N^2 (\log N)^p,$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c, \alpha} \setminus \Omega_N$ there exists a set $\mathcal{E}_{N, \omega}$, with

$$\text{mes}(\mathcal{E}_{N, \omega}) \lesssim (\log \log N)^{-1/10}, \text{compl}(\mathcal{E}_{N, \omega}) \lesssim N (\log N)^6,$$

such that for any $x \in \mathbb{T}$ and any integer m , $(\log N)^{6A} \leq |m| \leq N$, we have

$$\text{dist}\left(\mathcal{E}^0 \cap \text{spec}\left(H^{(l_1)}(x, \omega) \setminus \mathcal{E}_{N, \omega}\right), \text{spec}\left(H^{(l_2)}(x + m\omega, \omega)\right)\right) \geq \frac{2}{N (\log N)^p},$$

$l_1, l_2 \in \{l, l+1, 2l, 2l+1\}$.

Proof. Let $\Omega_N^1, \mathcal{E}_{N,\omega}^1$ denote the sets $\Omega_N, \tilde{\mathcal{E}}_{N,\omega}$ from Proposition 5.6. Let $\Omega_N^2, \mathcal{E}_{N,\omega}^2$ denote the sets $\Omega_N, \mathcal{E}_{N,\omega}$ from Proposition 7.6, with $\tilde{p} = p$. We define $\Omega_N = \Omega_l^1 \cup \Omega_{l+1}^1 \cup \Omega_{2l}^1 \cup \Omega_{2l+1}^1 \cup \Omega_N^2$ and

$$\mathcal{E}_{N,\omega} = \left\{ E \in \mathcal{E}^0 : \text{dist} \left(\mathcal{E}_{l,\omega}^1 \cup \mathcal{E}_{l+1,\omega}^1 \cup \mathcal{E}_{2l,\omega}^1 \cup \mathcal{E}_{2l+1,\omega}^1 \cup \mathcal{E}_{N,\omega}^2 \cup (\mathcal{E}^0)^C \right) \leq \frac{2}{N(\log N)^p} \right\}.$$

It is straightforward to check the measure and complexity bounds for Ω_N and $\mathcal{E}_{N,\omega}$.

To obtain the conclusion we argue by contradiction. Fix $x \in \mathbb{T}$ and $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \Omega_N$. Suppose there exist $l_1, l_2 \in \{l, l+1, 2l, 2l+1\}$, j_1, j_2 , and m , $|m| \geq (\log N)^{6A}$ such that $E_{j_1}^{(l_1)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N,\omega}$ and

$$\left| E_{j_1}^{(l_1)}(x, \omega) - E_{j_2}^{(l_2)}(x + m\omega, \omega) \right| < \frac{2}{N(\log N)^p}. \quad (7.11)$$

By the definition of $\mathcal{E}_{l,\omega}$ we have that $E_{j_i}^{(l_i)}(x, \omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{l_i,\omega}^1$, $i = 1, 2$. We can apply Proposition 5.6 to conclude that there exist eigenvalues $E_{k_1}^{(l')}(x + n_1\omega, \omega)$, $n_1 \in [0, l_1 - 1]$ and $E_{k_2}^{(l')}(x + n_2\omega, \omega)$, $n_2 \in [m, m + l_2 - 1]$, $l' = 100[(\log N)/\gamma]$, such that

$$\left| E_{j_1}^{(l_1)}(x, \omega) - E_{k_1}^{(l')}(x + n_1\omega, \omega) \right| \lesssim \exp(-\gamma l'/3) < 1/N^2, \quad (7.12)$$

$$\left| E_{j_2}^{(l_2)}(x + m\omega, \omega) - E_{k_2}^{(l')}(x + n_2\omega, \omega) \right| \lesssim \exp(-\gamma l'/3) < 1/N^2. \quad (7.13)$$

By the definition of $\mathcal{E}_{N,\omega}$ we have $E_{k_1}(x + n_1\omega) \in \mathcal{E}^0 \setminus \mathcal{E}_{N,\omega}^2$. We can apply Proposition 7.6, with $\tilde{p} = p$, to get

$$\left| E_{k_1}^{(l')}(x + n_1\omega, \omega) - E_{k_2}^{(l')}(x + n_2\omega, \omega) \right| \geq \frac{3}{N(\log N)^p}.$$

The above inequality, together with (7.12), and (7.13) contradicts (7.11). This concludes the proof. \square

Finally we obtain our main result.

Theorem 7.8. Fix $p > 15$. There exists a constant $N_0 = N_0(\|a\|_\infty, \|b\|_*, c, \alpha, \gamma, E^0, p)$ such that for any $N \geq N_0$ there exists a set $\tilde{\Omega}_N$, with

$$\text{mes}(\tilde{\Omega}_N) \lesssim \exp(-(\log \log \log N)^2/20), \text{compl}(\tilde{\Omega}_N) \lesssim N^2 (\log N)^p,$$

such that for any $\omega \in \Omega^0 \cap \mathbb{T}_{c,\alpha} \setminus \tilde{\Omega}_N$ there exists a set $\tilde{\mathcal{E}}_{N,\omega}$, with

$$\text{mes}(\tilde{\mathcal{E}}_{N,\omega}) \lesssim (\log \log N)^{-1/10}, \text{compl}(\tilde{\mathcal{E}}_{N,\omega}) \lesssim N (\log N)^6,$$

such that for any $x \in \mathbb{T}$, if $E_j^{(N)}(x, \omega) \in \mathcal{E}^0 \setminus \tilde{\mathcal{E}}_{N,\omega}$, for some j , then

$$\left| E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega) \right| \geq \frac{1}{N(\log N)^p},$$

for any $k \neq j$.

Proof. We start by identifying the parameters from the Elimination Assumption 3.1. Apply Proposition 7.3 with $A = A(\alpha)$ as in the Elimination Assumption 3.1. We can choose $\Omega_N, \mathcal{E}_{N,\omega}$ as in Proposition 7.7 and we also have $Q_N = (\log N)^{6A}$, $\sigma_N = 2N^{-1}(\log N)^{-p}$.

Next we apply Theorem 4.4 with $N' = \exp((\log N)^{1/7A})$. The conclusion follows by setting $\tilde{\Omega}_N = \Omega_N \cup \Omega_{N'}$, and

$$\tilde{\mathcal{E}}_{N,\omega} = \left\{ E \in \mathcal{E}^0 : \text{dist} \left(\mathcal{E}_{N,\omega} \cup \mathcal{E}_{N',\omega} \cup (\mathcal{E}^0)^C \right) < 2N^{-1}(\log N)^{-p} \right\}.$$

□

A Appendix

In this section we discuss how to obtain some of the results stated in Section 2 from the results of [BV12].

We start by discussing the large deviations estimate for determinants as stated in Proposition 2.1. For convenience we recall three relevant results from [BV12]. Note that in what follows the assumption $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$, $L(\omega, E) > \gamma > 0$ is implicit. Also, we use the notation $\langle \log |f_n^a| \rangle = \int_{\mathbb{T}} \log |f_n^a(x)| dx$ and $I_{a,E} = \int_{\mathbb{T}} \log |a(x) - E| dx$.

Proposition A.1. ([BV12, Proposition 4.10]) *There exist constants $c_0 = c_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$, $C_0 = C_0(\omega) > \alpha + 2$, and $C_1 = C_1(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that for every integer $n > 1$ and any $\delta > 0$ we have*

$$\text{mes} \{ x \in \mathbb{T} : |\log |f_n^a(x)| - \langle \log |f_n^a| \rangle| > n\delta \} \leq C_1 \exp \left(-c_0 \delta n (\log n)^{-C_0} \right).$$

Lemma A.2. ([BV12, Lemma 4.11]) *There exists a constant $C_0 = C_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that*

$$|\langle \log |f_n^a| \rangle - nL_n^a| \leq C_0$$

for all integers.

Lemma A.3. ([BV12, Lemma 3.9]) *For any integer $n > 1$ we have*

$$0 \leq L_n - L = L_n^u - L^u = L_n^a - L^a < C_0 \frac{(\log n)^2}{n}$$

where $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma)$.

Proposition 2.1 is a straightforward consequence of the above results. Note that the constants depend on ω rather than c, α as in Section 2. However, in [BV12] it was noted that the dependence on ω only comes through the large deviations estimate for subharmonic functions [GS01, Theorem 3.8]. The dependence there is only on c, α , so we can replace ω with c, α . The dependence of the constants on $I_{a,E}$ in [BV12] came through [BV12, Lemma 4.2]. We provide a different proof of this lemma that gets rid of the dependence on $I_{a,E}$.

First we need to recall three results that will be needed for the proof. The following theorem is a restatement of the large deviations estimate for subharmonic functions, [GS01, Theorem 3.8]. In what follows \mathcal{A}_ρ denotes the annulus $\{z : |z| \in (1 - \rho, 1 + \rho)\}$.

Theorem A.4. ([GS01, Theorem 3.8]) Fix $p > \alpha + 2$. Let u be a subharmonic function and let

$$u(z) = \int_{\mathbb{C}} \log|z - \zeta| d\mu(\zeta) + h(z)$$

be its Riesz representation on a neighborhood of \mathcal{A}_ρ . If $\mu(\mathcal{A}_\rho) + \|h\|_{L^\infty(\mathcal{A}_\rho)} \leq M$ then for any $\delta > 0$ and any positive integer n we have

$$\text{mes} \left(\left\{ x \in \mathbb{T} : \left| \sum_{k=1}^n u(x + k\omega) - n \langle u \rangle \right| > \delta n \right\} \right) < \exp(-c_0 \delta n + r_n)$$

where $c_0 = c_0(c, \alpha, M, \rho)$ and

$$r_n = \begin{cases} C_0 (\log n)^p & , n > 1 \\ C_0 & , n = 1, \end{cases}$$

with $C_0 = C_0(c, \alpha, p)$. If p_s/q_s is a convergent of ω and $n = q_s > 1$ then one can choose $r_n = C_0 \log n$.

Proposition A.5. ([BV12, Theorem 3.10]) Fix $p > \alpha + 2$. For any $\delta > 0$ and any integer $n > 1$ we have

$$\text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - n L_n^a| > \delta n\} < \exp(-c_0 \delta n + C_0 (\log n)^p)$$

where $c_0 = c_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$ and $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma, p)$. The same estimate, with possibly different constants, holds with L^a instead of L_n^a .

Lemma A.6. ([GS08, Lemma 2.4]) Let u be a subharmonic function defined on \mathcal{A}_ρ such that $\sup_{\mathcal{A}_\rho} u \leq M$. There exist constants $C_1 = C_1(\rho)$ and C_2 such that, if for some $0 < \delta < 1$ and some L we have

$$\text{mes} \{x \in \mathbb{T} : u(x) < -L\} > \delta,$$

then

$$\sup_{\mathbb{T}} u \leq C_1 M - \frac{L}{C_1 \log(C_2/\delta)}.$$

We can now reprove [BV12, Lemma 4.2]. Analogously to f_N^a and f_N^u , f_N will be the top left entry in M_N . From (2.1) it follows that

$$f_N(z) = \left(\prod_{j=1}^N b(z + j\omega) \right)^{-1} f_N^a(z). \quad (\text{A.1})$$

Lemma A.7. (cf. [BV12, Lemma 4.2]) Let $(\omega, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ be such that $L(\omega, E) > \gamma > 0$. There exists $l_0 = l_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$ such that

$$\text{mes} \{x \in \mathbb{T} : |f_l(x)| \leq \exp(-l^3)\} \leq \exp(-l)$$

for all $l \geq l_0$.

Proof. We argue by contradiction. Assume

$$\text{mes}\{x \in \mathbb{T} : |f_l(x)| \leq \exp(-l^3)\} > \exp(-l)$$

for some sufficiently large l . We have that

$$|f_l^a(x)| = |f_l(x)| \prod_{j=1}^l |b(x+j\omega)| \leq \exp(-l^3) C^l \leq \exp(-l^3/2)$$

on a set of measure greater than $\exp(-l)$. Hence

$$\text{mes}\{x \in \mathbb{T} : |f_l^a(x)| \leq \exp(-l^3/2)\} > \exp(-l).$$

At the same time we have

$$\sup_{\mathbb{T}} \log |f_l^a| \leq \sup_{\mathbb{T}} \log \|M_l^a\| \leq Cl,$$

so by applying Lemma A.6 we get

$$\sup_{\mathbb{T}} |f_l^a| \leq \exp\left(C_1 l - \frac{l^3}{C_2 \log(C_3 \exp(l))}\right) \leq \exp(-Cl^2). \quad (\text{A.2})$$

Using Proposition A.5 and (2.2) (recall that $\tilde{b} = \bar{b}$ on \mathbb{T}) we get

$$\begin{aligned} \exp(lL^a - l^{1/3}) \leq \|M_l^a(x)\| \leq & \left(|f_l^a(x)|^2 + |b(x+l\omega)f_{l-1}^a(x)|^2 \right. \\ & \left. + |b(x)f_{l-1}^a(x+\omega)|^2 + |b(x)b(x+l\omega)f_{l-2}^a(x+\omega)|^2 \right)^{1/2} \end{aligned} \quad (\text{A.3})$$

for all x except for a set of measure less than $\exp(-c_1 l^{1/3} + C(\log l)^p) < \exp(-cl^{1/3})$. Our plan is to contradict (A.3) by showing that

$$\begin{aligned} |f_l^a(x)|^2 + |b(x+l\omega)f_{l-1}^a(x)|^2 + |b(x)f_{l-1}^a(x+\omega)|^2 \\ + |b(x)b(x+l\omega)f_{l-2}^a(x+\omega)|^2 < \exp(2lL^a - 2l^{1/3}), \end{aligned} \quad (\text{A.4})$$

for x in some set of measure much larger than $\exp(-cl^{1/3})$. The first term is already taken care of by (A.2). We will show that we can provide a convenient upper bound for the next two terms when x is in some set of measure much larger than $\exp(-cl^{1/3})$. For this we argue again by contradiction. Suppose

$$|f_{l-1}^a(x)| \geq \exp(lL^a - l^{1/2}) \quad (\text{A.5})$$

for $x \in G$, with $\text{mes}(G) \geq 1/2 - l^{-2}$. Using Corollary 2.3 we can apply Cartan's estimate Lemma 2.9 (with $H = l^{1/4}$) to $\log |f_{l-1}^a(\cdot)|$ on $\mathcal{D}(x_0, l^{-1})$, for any $x_0 \in G$, to get

$$|f_{l-1}^a(x)| \geq \exp(lL^a - l^{5/6}), \quad (\text{A.6})$$

for $x \in \mathcal{D}(x_0, l^{-1}/6) \setminus \mathcal{B}_{x_0}$, $\text{mes}(\mathcal{B}_{x_0}) \leq \exp(-l^{1/4})$. It is straightforward to see that (A.6) holds on a set $G' \supset G$, with $\text{mes}(G') \geq 1/2 + cl^{-1}$. Hence, we have

$$|f_{l-1}^a(x)|, |f_{l-1}^a(x+\omega)| \geq \exp(lL^a - l^{5/6}), \quad (\text{A.7})$$

on the set $G'' = G' \cap (G' + \omega)$, with $\text{mes}(G'') > cl^{-1}$. Let $P_l(x, \omega) = \prod_{j=0}^{l-1} b(x + j\omega)$. We will obtain a contradiction by using the identity

$$\begin{aligned} \overline{P_l(x, \omega)} P_l(x + \omega, \omega) &= \det M_l^a(x) = -\overline{b(x)} b(x + l\omega) f_l^a(x) f_{l-2}^a(x + \omega) \\ &\quad + \overline{b(x)} b(x + l\omega) f_{l-1}^a(x) f_{l-1}^a(x + \omega). \end{aligned}$$

Indeed, from the above identity it follows that

$$\overline{P_{l-1}(x + \omega, \omega)} P_{l-1}(x + \omega, \omega) = -f_l^a(x) f_{l-2}^a(x + \omega) + f_{l-1}^a(x) f_{l-1}^a(x + \omega),$$

and hence

$$\frac{|f_l^a(x) f_{l-2}^a(x + \omega)|}{|P_{l-1}(x + \omega, \omega)|^2} \geq \frac{|f_{l-1}^a(x) f_{l-1}^a(x + \omega)|}{|P_{l-1}(x + \omega, \omega)|^2} - 1. \quad (\text{A.8})$$

From Theorem A.4 it follows that

$$\exp(lD - l^{1/2}) \leq |P_{l-1}(x + \omega, \omega)| \leq \exp(lD + l^{1/2}),$$

for $x \in \mathbb{T} \setminus \mathcal{B}$, with $\text{mes}(\mathcal{B}) < \exp(-l^{1/3})$. On one hand we have

$$\frac{|f_l^a(x) f_{l-2}^a(x + \omega)|}{|P_{l-1}(x + \omega, \omega)|^2} \leq \exp(-Cl^2 + lL^a + (\log l)^C - 2lD + 2l^{1/2}) \leq \exp(-cl^2),$$

for $x \in \mathbb{T} \setminus \mathcal{B}$. On the other hand, for $x \in G'' \setminus \mathcal{B}$ we have

$$\begin{aligned} \frac{|f_{l-1}^a(x) f_{l-1}^a(x + \omega)|}{|P_{l-1}(x, \omega)|^2} &\geq \exp(2lL^a - 2l^{5/6} - 2lD - 2l^{1/2}) \\ &= \exp(2lL - 2l^{5/6} - 2l^{1/2}) \geq \exp(2\gamma l - 4l^{5/6}). \end{aligned}$$

Since $G'' \setminus \mathcal{B} \neq \emptyset$, the previous two inequalities contradict (A.8). Hence we must have $\text{mes}(G) < 1/2 - l^{-2}$. In other words we have

$$|f_{l-1}^a(x)| < \exp(lL^a - l^{1/2}),$$

for $x \in B = \mathbb{T} \setminus G$, $\text{mes}(B) \geq 1/2 + l^{-2}$. It follows that

$$|f_{l-1}^a(x)|, |f_{l-1}^a(x + \omega)| < \exp(lL^a - l^{1/2}), \quad (\text{A.9})$$

for $x \in B' = B \cap (B + \omega)$, $\text{mes}(B') > l^{-2}$.

By writing

$$M_l^a(x - \omega) = M_l^a(x) \begin{bmatrix} a(x - \omega) - E & -\overline{b(x - \omega)} \\ b(x) & 0 \end{bmatrix},$$

we get

$$f_l^a(x - \omega) = (a(x - \omega) - E)f_{l-1}^a(x) - |b(x)|^2 f_{l-2}^a(x + \omega).$$

From this we get

$$\begin{aligned} |b(x)b(x + l\omega)f_{l-2}^a(x + \omega)| &= \frac{|b(x + l\omega)|}{|b(x)|} |f_l^a(x - \omega) - (a(x - \omega) - E)f_{l-1}^a(x)| \\ &\leq C \exp(-D + l^{1/3}) (\exp(-Cl^2) + C \exp(lL^a - l^{1/2})) \leq C \exp(lL^a - l^{1/2}/2), \end{aligned} \quad (\text{A.10})$$

for $x \in B'' \subset B'$, $\text{mes}(B'') > l^{-2} - \exp(-l^{1/4}) > l^{-3}$ (note that we used Theorem A.4). By (A.2), (A.9), and (A.10) we have that (A.4) holds for $x \in B''$. Since $\text{mes}(B'') \gg \exp(-cl^{1/3})$, this contradicts (A.3), as desired, and concludes the proof. \square

Next we discuss the uniform upper bound result, Proposition 2.2, and its consequences, Corollary 2.3, Corollary 2.4, as well as Lemma 2.5. For convenience we state the uniform upper bound result from [BV12].

Proposition A.8. ([BV12, Proposition 3.14]) *Fix $p > \alpha + 2$. For any integer $n > 1$ we have that*

$$\sup_{x \in \mathbb{T}} \log \|M_n^a(x)\| \leq nL_n^a + C_0(\log n)^p$$

where $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma, p)$.

Proposition 2.2 follows immediately from the above proposition and Lemma A.3.

Next we recall some further results needed to prove Corollary 2.3. The statement of the results is adapted to our setting.

Lemma A.9. ([BV12, Corollary 3.13]) *There exists a constant $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \rho_0)$ such that*

$$|L_n^u(y_1) - L_n^u(y_2)| = |L_n^a(y_1) - L_n^a(y_2)| \leq C_0 |r_1 - r_2|$$

for any $y_1, y_2 \in (1 - \rho_0, 1 + \rho_0)$ and any positive integer n .

Lemma A.10. ([BV12, Corollary 3.17]) *Let $(\omega, E_0) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega, E_0) > \gamma > 0$. There exist constants $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$, $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$, and $n_0 = n_0(\|a\|_\infty, \|b\|_*, |E_0|, c, \alpha, \gamma)$ such that we have*

$$|n(L_n(\omega, E) - L_n(\omega, E_0))| = |n(L_n^a(\omega, E) - L_n^a(\omega, E_0))| \leq n^{-C_0}$$

for $n \geq n_0$ and $|E - E_0| < n^{-C_1}$.

A straightforward replacement of E with ω in the proof of [BV12, Corollary 3.17] yields the following result.

Lemma A.11. *Let $(\omega_0, E) \in \mathbb{T}_{c,\alpha} \times \mathbb{C}$ such that $L(\omega_0, E) > \gamma > 0$. There exist constants $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$, $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$, $n_0 = n_0(\|a\|_\infty, \|b\|_*, |E|, c, \alpha, \gamma)$ such that we have*

$$|n(L_n(\omega, E) - L_n(\omega_0, E))| = |n(L_n^a(\omega, E) - L_n^a(\omega_0, E))| \leq n^{-C_0}$$

for $n \geq n_0$ and $|\omega - \omega_0| < n^{-C_1}$.

We have all we need to prove Corollary 2.3.

Proof. (of Corollary 2.3) From Lemma A.3, Lemma A.9, Lemma A.10, Lemma A.11 it is straightforward to conclude that there exists $\rho \ll \rho_0$ such that $L(y, \omega, E) > \gamma/2$ for $|y|, |\omega - \omega_0|, |E - E_0| \leq \rho$. We can apply Proposition 2.2 to get

$$\sup_{x \in \mathbb{T}} \log \|M_N^a(x + iy, \omega, E)\| \leq NL^a(y, \omega, E) + C(\log N)^{C'},$$

for $|y|, |\omega - \omega_0|, |E - E_0| \leq \rho$. The conclusion follows from Lemma A.9, Lemma A.10, and Lemma A.11. \square

Based on the already established results it is straightforward to see that Corollary 2.4 follows with the same proof as [GS11, Corollary 2.15]. For the sake of completeness we include the proof.

Proof. (of Corollary 2.4) Let ∂ denote any of the partial derivatives $\partial_x, \partial_y, \partial_\omega, \partial_E$. We have

$$\begin{aligned} \partial M_N^a(x + iy, \omega, E) &= \sum_{j=0}^{N-1} M_{N-j-1}^a(x + iy + (j+1)\omega, \omega, E) \\ &\quad \cdot \partial \begin{bmatrix} a(x + iy + j\omega) - E & -\tilde{b}(x + iy + j\omega) \\ -b(x + iy + (j+1)\omega) & 0 \end{bmatrix} M_j^a(x + iy, \omega, E). \end{aligned}$$

The estimate (2.5) follows from Corollary 2.3 by using the above identity and the mean value theorem.

From (2.5) it follows that

$$\begin{aligned} &|f_N^a(x + iy, \omega, E) - f_N^a(x_0, \omega_0, E_0)| \\ &\leq (|E - E_0| + |\omega - \omega_0| + |x - x_0| + |y|) \exp\left(NL^a(\omega_0, E_0) + (\log N)^C\right). \end{aligned}$$

The estimate (2.6) follows by dividing both sides by $|f_N^a(x_0, \omega_0, E_0)|$ and by using the fact that $|\log x| \lesssim |x - 1|$ for $x \in (1/2, 3/2)$. \square

Finally, Lemma 2.5 can be proved along the same lines as [BV12, Proposition 3.14] and Corollary 2.3. We also need to recall the following result.

Lemma A.12. ([GS08, Lemma 4.1]) *Let u be a subharmonic function and let*

$$u(z) = \int_{\mathbb{C}} \log|z - \zeta| d\mu(\zeta) + h(z)$$

be its Riesz representation on a neighborhood of \mathcal{A}_ρ . If $\mu(\mathcal{A}_\rho) + \|h\|_{L^\infty(\mathcal{A}_\rho)} \leq M$ then for any $r_1, r_2 \in (1 - \rho, 1 + \rho)$ we have

$$|\langle u(r_1(\cdot)) \rangle - \langle u(r_2(\cdot)) \rangle| \leq C_0 |r_1 - r_2|,$$

where $C_0 = C_0(M, \rho)$.

Proof. (of Lemma 2.5) It is enough to prove the estimate for S_N . Note that $\|b\|_* = \|\tilde{b}\|_*$.

We first prove the uniform upper bound with $y = 0$. It is sufficient to establish the estimate for large N . Fix $p > \alpha + 2$. From Theorem A.4 we have

$$S_N(x + iy, \omega) - ND(y) \leq C(\log n)^p \quad (\text{A.11})$$

except for a set $\mathcal{B}(y)$ of measure less than

$$\exp(-c_1 C(\log n)^p + C'(\log n)^p) < \exp(-c(\log n)^p).$$

By the subharmonicity of S_N we have

$$\begin{aligned} S_N(x_0, \omega) - ND &\leq \frac{1}{\pi N^{-2}} \int_{\mathcal{D}(x_0, N^{-1})} (S_N(z, \omega) - ND) dA(z) \\ &\leq \frac{1}{\pi N^{-2}} \int_{-N^{-1}}^{N^{-1}} \int_{x_0 - N^{-1}}^{x_0 + N^{-1}} |S_N(x + iy, \omega) - ND| dx dy. \end{aligned} \quad (\text{A.12})$$

For $y \in (-N^{-1}, N^{-1})$, by using (A.11) and Lemma A.9, we have

$$\begin{aligned} &\int_{x_0 - N^{-1}}^{x_0 + N^{-1}} |S_N(x + iy, \omega) - ND| dx \\ &\leq \int_{x_0 - N^{-1}}^{x_0 + N^{-1}} |S_N(x + iy, \omega) - ND(y)| dx + 2|D - D(y)| \\ &\leq C(\log N)^p N^{-1} + CN \exp(-c(\log N)^p / 2) + CN^{-1} \leq C(\log N)^p N^{-1}. \end{aligned}$$

We used the fact that $\|S_N(\cdot, \omega) - ND\|_{L^2(\mathbb{T})} \leq CN$ (a straightforward consequence of Theorem A.4) to deal with the exceptional set $\mathcal{B}(y)$. Plugging this estimate in (A.12) yields that

$$\sup_{x \in \mathbb{T}} S_N(x, \omega) \leq ND + C(\log N)^p,$$

with $C = C(\|b\|_*, c, \alpha)$. This yields (by replacing b with $b(\cdot + iy)$) that

$$\sup_{x \in \mathbb{T}} S_N(x + iy, \omega) \leq ND(y) + C(\log N)^p.$$

The conclusion follows from Lemma A.12. \square

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